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TIME-RESPONSE FUNCTIONS OF INERTOVISCOELASTIC NETWORKS

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ABSTRACT

This paper derives the causal time-response functions of three-parameter mechanical networks that have been reported in the literature and involve the inerter—a two-node element in which the force-output is proportional to the relative acceleration of its end-nodes. It is shown that all frequency-response functions that exhibit singularities along the real frequency axis need to be enhanced with the addition of a Dirac delta function or with its derivative depending on the strength of the singularity. In this way the real and imaginary parts of the enhanced frequency response functions are Hilbert pairs; therefore, yielding a causal time-response function in the time domain.
Keywords: Analytic functions; causality; electrical-mechanical analogies; mechanical networks; seismic protection; suspension systems; vibration absorption

1. INTRODUCTION

The force-current; and therefore, velocity-voltage analogy between mechanical and electrical networks (Firestone 1933) respects the in-series and in-parallel configuration of connections, so that equivalent mechanical and electrical networks are expressed by similar diagrams. According to the force-current/velocity-voltage analogy the elastic spring corresponds to the inductor and the linear dashpot corresponds to the resistor. In an effort to lift the constraint that a lumped mass element in a mechanical network has always one of its end-nodes (terminals) connected to the ground, Smith (2002) proposed a linear mechanical element that he coined "the inerter" in which the output force is proportional only to the relative acceleration between its end-nodes. Accordingly, the inerter is the precise mechanical analogue of the capacitor. For instance, the driving spinning-top shown in Figure 1 is a physical realization of the inerter given that the driving force is only proportional to the relative acceleration between terminals 1 and 2. The constant of proportionality of the inerter is coined the "inertance"=M_R (Smith 2002) and has units of mass [M]. The unique characteristic of the inerter is that it has an appreciable inertial mass as oppose to a marginal gravitational mass. Accordingly, if F_1, u_1 and F_2, u_2 are the forces and displacements at the end-nodes of the inerter with inertance M_R, its constitutive relation is defined as:

$$\begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix} = \begin{bmatrix} M_R & -M_R \\ -M_R & M_R \end{bmatrix} \begin{Bmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \end{Bmatrix} \quad (1)$$

In (1), the force $F_1(t)$=-$F_2(t)$=$M_R(\ddot{u}_1(t)$-$\ddot{u}_2(t))$ is the through variable of the inerter; whereas, the absolute displacements u_1 (respectively $\ddot{u}_1$) and u_2 (respectively $\ddot{u}_2$) are the across variables. Smith and his coworkers developed and tested both a rack-and-pinion inerter and a ball-screw inerter (Papageorgiou and Smith 2005, Papageorgiou et al. 2008). Upon its conceptual development and experimental validation, the inerter was implemented to control the suspension vibrations of racing cars under the name of *J-damper* (Chen et al. 2009, Kuznetsov 2010). About the same time a two-terminal flywheel was proposed for the suppression of vehicle vibrations (Li et al. 2011).
In parallel with the aforementioned developments in vehicle mechanics and dynamics, during the last

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Figure 1. A physical realization of the inerter which is the mechanical analogue of the capacitor in a force-current/velocity-voltage analogy.

decade a growing number of publications have proposed the use of rotational inertia dampers for the vibration control and seismic protection of civil structures. For instance, (Hwang et al. 2007) proposed a rotational inertia damper in association with a toggle bracing for vibration control of building structures. The proposed rotational inertia damper consists of a cylindrical mass that is driven by a ball screw and rotates within a chamber that contains some viscous fluid. In this way the vibration reduction originates partly from the difficulty to mobilize the rotational inertia of the rotating mass and partly from the difficulty to shear of the viscous fluid that surrounds the rotating mass. The use of inerters to improve the performance of seismic isolated buildings has been proposed in (Wang et al. 2007); while, (Ikago et al. 2012) examined the dynamic response of a single-degree-of-freedom (SDOF) structure equipped with a rotational damper that is very similar to the rotational inertia damper initially proposed in (Hwang et al. 2007). The main difference is that, in the configuration proposed in (Ikago et al. 2012), an additional flywheel is appended to accentuate the rotational inertia effect of the proposed vibration control device. About the same time, (Takewaki et al. 2012) examined the response of SDOF and multi-degree-of-freedom (MDOF) structures equipped with supplemental rotational inertia that is offered from a ballscrew type device that sets in motion a rotating flywheel. Subsequent studies on the response of MDOF structures equipped with supplemental rotational inertia have been presented by (Marian and Giaralis 2014, Lazar et al. 2014, Giaralis and Taflanidis 2015) within the context of enhancing the performance of tuned mass dampers. More recently, (Makris and Kampas 2016) showed that the seismic protection of structures with supplemental rotational inertia has some unique advantages, particularly in suppressing the spectral displacement of long period structures—a function that is not efficiently achieved with large values of supplemental damping. However, this happens at the expense of transferring appreciable forces at the support of the flywheels (chevron frames for buildings or end-abutments for bridges).

One of the challenges with the dynamic response analysis of civil structures is that while the inerter, or more complex response-modification mechanical networks that involve inerters, are linear networks, the overall structural system in which they belong may behave nonlinearly. In this case the overall structural response needs to be computed in the time-domain. A time-domain representation of the response modification network is possible either via a state-space formulation; or by computing the basic time-response function of the response-modification network and proceeding by solving a set of integro-differential equations. Given that the state-space formulation of some mechanical networks that contain inerters involve the evaluation of the third derivative of the end-node displacement (derivative of the end-node acceleration, see (Makris and Kampas 2016) and equations (2) and (3) of this paper), the alternative of calculating the response-history of the through or across variables of the mechanical network by convolving its basic time-response functions becomes attractive.

2. MOTIVATION AND PROBLEM STATEMENT

Given that the inerter, as defined with (1), complements the linear spring and the viscous dashpot as the third elementary response-modification element, this paper examines the time-response functions

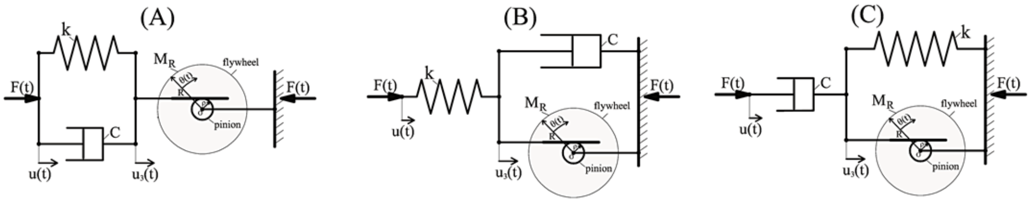

Figure 2. The three-parameter inertoviscoelastic fluid A, B and C.

of the three-parameter inertoviscoelastic "fluid" networks shown in Fig. 2. The term "fluid" expresses that the network undergoes an infinite displacement under static loading.

Figure 2 (A) is a spring-dashpot parallel connection (Kelvin-Voight model) that is connected in series with an inerter. This mechanical network, that is coined the inertoviscoelastic fluid A, emerged during the testing of inerters where the spring-dashpot parallel connection served as a mechanical buffer between a prototype inerter and the driving actuator (Papageorfiou and Smith 2005). The effectiveness of the inertoviscoelastic fluid A was subsequently studied extensively by (Lazar et al. 2014) in comparison with the traditional tuned-mass-damper that finds applications in the reduction of building vibrations; whereas, (Makris and Kampas 2016) used the same three-parameter model to study the effectiveness of an inerter mounted on a chevron frame for the seismic protection of buildings (or on a bridge abutment for the seismic protection of bridges) with finite stiffness (spring) and damping (dashpot). The constitutive equation of the three-parameter mechanical network shown in Figure 2 (A) is described in (Makris and Kampas 2016),

$$\frac{k}{M_R}F(t)+\frac{C}{M_R}\frac{dF(t)}{dt}+\frac{d^2F(t)}{dt^2}=k\frac{d^2u(t)}{dt^2}+C\frac{d^3u(t)}{dt^3} \tag{2}$$

By defining the relaxation time, $\lambda=C/k$ and the rotational frequency $\omega_R=\sqrt{k/M_R}$ (2) assumes the form:

$$F(t)+\lambda\frac{dF(t)}{dt}+\frac{1}{\omega_R^2}\frac{d^2F(t)}{dt^2}=M_R\left(\frac{d^2u(t)}{dt^2}+\lambda\frac{d^3u(t)}{dt^3}\right) \tag{3}$$

The right-hand-side (rhs) of the constitutive equation given by (2) or (3) involves the third derivative of the nodal displacement (derivative of the nodal acceleration) and this may challenge the accuracy of the numerically computed response, in particular when the input excitation is only available in digital form as in the case of recorded seismic accelerograms. Part of the motivation of this paper is to bypass this challenge (numerical evaluation of $d^3u(t)/dt^3$) by studying the integral representations of the force, $F(t)$ (through variable), and the displacement, $u(t)$ (end-node variable) appearing in (2) or (3).

Upon deriving the time-response functions of the inertoviscoelastic fluid A, the paper proceeds by studying the time-response functions of the inertoviscoelastic fluid B shown in Fig. 2 (B) which is a dashpot-inerter parallel connection (rotational inertia damper) that is connected in series with a spring that approximates the finite stiffness of the mounting connections of a rotational inertia damper (Ikago et al. 2012). Next, the paper examines the time-response functions of the inertoviscoelastic fluid C shown in Figure 2 (C) which is a spring-inerter parallel connection (inertoelastic solid) that is connected in series with a dashpot. Again, the constitutive equation of the inertoviscoelastic fluid C involves the third derivative of the nodal displacements (derivative of the nodal accelerations) which may challenge the accuracy of the numerical calculation of a state-space formulation. Accordingly, the integral representation of the force and displacement presented in this study offers an attractive alternative.

3. FREQUENCY- AND TIME-RESPONSE FUNCTIONS

When a combination of springs, dashpots and inerters form a mechanical network, the constitutive equation of the mechanical network is of the form

$$\left[\sum_{m=0}^{M} a_m \frac{d^m}{dt^m}\right] F(t) = \left[\sum_{n=0}^{N} b_n \frac{d^n}{dt^n}\right] u(t) \tag{4}$$

where $F(t)$ is the force (through variable) and $u(t)$ is the relative displacement of its end-nodes. In (4) the coefficients a_m and b_n are restricted to real numbers and the order of differentiation m and n is restricted to integers. The linearity of (4) permits its transformation in the frequency domain by applying the Fourier transform

$$u(\omega) = H(\omega)F(\omega) = \left[H_1(\omega) + iH_2(\omega)\right]F(\omega) \tag{5}$$

where $u(\omega) = \int_{-\infty}^{\infty} u(t)e^{-i\omega t}dt$ and $F(\omega) = \int_{-\infty}^{\infty} F(t)e^{-i\omega t}dt$ are the Fourier transforms of the relative displacement and force histories respectively; and $H(\omega)$ is the dynamic compliance (dynamic flexibility) of the network:

$$H(\omega) = \frac{u(\omega)}{F(\omega)} = \frac{\sum_{m=0}^{M} a_m (i\omega)^m}{\sum_{n=0}^{N} b_n (i\omega)^n} \tag{6}$$

The dynamic compliance of a mechanical network, $H(\omega)$, as expressed by (5) is a transfer function that relates a force input to a displacement output. When the dynamic compliance $H(\omega)$ is a proper transfer function, the relative displacement, $u(t)$, in (4) can be computed in the time domain via the convolution

$$u(t) = \int_{-\infty}^{t} h(t-\tau)F(\tau)d\tau \tag{7}$$

where $h(t)$ is the "impulse response function" defined as the resulting displacement at time t for an impulsive force input at time τ ($\tau<t$) and is the inverse Fourier transform of the dynamic compliance

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{i\omega t}d\omega \tag{8}$$

The mechanical impedance, $Z(\omega)=Z_1(\omega)+iZ_2(\omega)$, is a transfer function which relates a velocity input to a force output

$$F(\omega) = Z(\omega)v(\omega) = \left[Z_1(\omega) + iZ_2(\omega)\right]v(\omega) \tag{9}$$

where $v(\omega)=i\omega u(\omega)$ is the Fourier transform of the relative velocity time-history. The classical definition of the mechanical impedance as expressed by (9) (Morse and Feshbach 1953, Harris and Crede 1976 among others) is adopted in this paper given that its corresponding time-response function, known as the relaxation stiffness, $k(t)$ (see (12)), is a most practical time-response function which can be measured experimentally with a simple relaxation test. Accordingly, for the linear inertoviscoelastic model given by (4), the mechanical impedance is

$$Z(\omega) = \frac{F(\omega)}{v(\omega)} = \frac{\sum_{n=0}^{N} b_n (i\omega)^n}{\sum_{m=0}^{M} a_m (i\omega)^{m+1}} \tag{10}$$

Smith (2002) adopts as definition of the mechanical impedance the inverse of the classical definition

expressed by (10) in order to maintain the analogy with electrical engineering where the impedance is the ratio of the voltage across variable (here velocity) to the current though variable (here force).
The force output, $F(t)$ appearing in (4) can be computed in the time domain with the convolution integral

$$F(t)=\int_{-\infty}^{t} k(t-\tau)\dot{u}(\tau)d\tau \tag{11}$$

where $k(t)$ is the relaxation stiffness of the mechanical network defined as the resulting force at the present time, t, due to a unit step-displacement input at time τ ($\tau<t$) and is the inverse Fourier transform of the impedance

$$k(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} Z(\omega)e^{i\omega t}d\omega \tag{12}$$

The inverse of the impedance, $Y(\omega)=1/Z(\omega)$, is the admittance; while in the mechanical and structural engineering literature the term "mobility" is used (Harris and Crede 1976). The admittance (mobility) is a transfer function that relates a force input to a velocity output and when is a proper transfer function, the relative velocity history between the end-nodes of the mechanical network can be computed in the time-domain via the convolution

$$v(t)=\int_{-\infty}^{t} y(t-\tau)F(\tau)d\tau \tag{13}$$

where $y(t)$ is the "impulse velocity response function" defined as the resulting velocity at time t for an impulsive force input at time τ ($\tau<t$) and is the inverse Fourier transform of the admittance:

$$y(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)e^{i\omega t}d\omega \tag{14}$$

At negative times ($t<0$), all three time-response functions given by Eqs. (8), (12) and (14) need to be zero in order for the phenomenological model (mechanical network) to be causal. The requirement for a time-response function to be causal in the time domain implies that its corresponding frequency-response function is analytic on the bottom-half complex plane (Bendat and Piersol 1986, Papoulis 1987, Bracewell 1986, Makris 1997a and Makris 1997b). The analyticity condition on a complex function, $Z(\omega)=Z_1(\omega)+iZ_2(\omega)$, relates the real part $Z_1(\omega)$ and the imaginary part $Z_2(\omega)$ with the Hilbert transform (Morse and Feshbach 1953, Bendat and Crede 1976, Papoulis 1987, Bracewell 1986 and Triverio et al. 2007) :

$$Z_1(\omega)=-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{Z_2(x)}{x-\omega}dx,\quad Z_2(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{Z_1(x)}{x-\omega}dx \tag{15}$$

Prior of computing the time-response functions of the three-parameter mechanical networks shown in Figs. 2 and 3, we first compute the time-response functions of the solitary inerter since its admittance and dynamic compliance exhibit singularities along the real frequency axis and need to be enhanced with either the addition of a Dirac delta function or with its derivative depending on the strength of the singularity.

4. FREQUENCY- AND TIME-RESPONSE FUNCTIONS OF THE INERTER

The first row of (1) gives:

$$F(t)=M_R\frac{d^2u(t)}{dt^2} \tag{16}$$

where $F(t)=F_1(t)=-F_2(t)$ is the through variable and $u(t)=u_1(t)-u_2(t)$ is the relative displacement of the end-nodes of the inerter. The Fourier transform of (16) is $F(\omega)=-M_R\omega^2u(\omega)$; therefore, the compliance of the inerter as defined by (6) is a proper transfer function:

$$H(\omega) = -\frac{1}{M_R}\frac{1}{\omega^2} \tag{17}$$

While the dynamic compliance (dynamic flexibility) of the inerter as expressed by (17) is a proper transfer function, the inverse Fourier transform of $-1/\omega^2$ is $(t/2)\cdot\mathrm{sgn}(t)$, where $\mathrm{sgn}(t)$ is the signum function. Accordingly, by using the expression of the compliance of the inerter as offered by (17), the resulting impulse response function as defined by (8) is $(M_R/2)t\cdot\mathrm{sgn}(t)$; which is clearly a non-causal function. In fact the signum function, $\mathrm{sgn}(t)$, indicates that there is as much response before the induced impulse force as the response upon the excitation is induced. Two decades ago, this impact was resolved (Bracewell 1986, Makris 1997a) by extending the relation between the analyticity of a transfer function and the causality of the corresponding time-response function to the case where generalized functions are involved(Bendat and Piersol 1986, Papoulis 1987 and Bracewell 1986). Given that the compliance of the inerter as expressed by (17) is a purely real quantity, we are in search of the imaginary Hilbert pair of $-1/\omega^2$.
The Hilbert pair of $-1/\omega^2$ is constructed by employing the first of equations (15), together with the property of the derivative of the Dirac delta function (Lighthill 1958):

$$\int_{-\infty}^{\infty}\frac{d\delta(t-0)}{dt}f(t)dt = -\int_{-\infty}^{\infty}\delta(t-0)\frac{df(t)}{dt}dt = -\frac{df(0)}{dt} \tag{18}$$

By letting $H_2(\omega) = \pi\frac{d\delta(\omega-0)}{d\omega}$, its Hilbert transform gives:

$$H_1(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\pi\frac{d\delta(x-0)}{dx}\frac{1}{x-\omega}dx \tag{19}$$

and with the change of variables $\xi = x-\omega$, $d\xi = dx$, (19) becomes:

$$\begin{aligned} H_1(\omega) &= -\int_{-\infty}^{\infty}\frac{d\delta(\xi-(-\omega))}{d\xi}\frac{1}{\xi}d\xi \\ &= \int_{-\infty}^{\infty}\delta(\xi-(-\omega))(-\frac{1}{\xi^2})d\xi = -\frac{1}{\omega^2} \end{aligned} \tag{20}$$

The result of (20) indicates that the rhs of (17) cannot stand alone and has to be accompanied by its imaginary Hilbert pair, $\pi d\delta(\omega-0)/d\omega$. Consequently, the correct expression of the dynamic compliance of the inerter is

$$H(\omega) = \frac{1}{M_R}\left[-\frac{1}{\omega^2} + i\pi\frac{d\delta(\omega-0)}{d\omega}\right] \tag{21}$$

By "manually" appending the imaginary part, $\pi d\delta(\omega-0)/d\omega$, in the rhs of (17), the inverse Fourier transform of the correct dynamic compliance of the inerter as expressed by (21) gives:

$$\begin{aligned} h(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}H(\omega)e^{i\omega t}d\omega \\ &= \frac{1}{M_R}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[-\frac{1}{\omega^2} + i\pi\frac{d\delta(\omega-0)}{d\omega}\right]e^{i\omega t}d\omega \end{aligned} \tag{22}$$

By recalling that the Fourier transform of $-1/\omega^2$ is $(t/2)\mathrm{sgn}(t)$, (22) simplifies to

$$h(t) = \frac{1}{M_R}\left[\frac{t}{2}\mathrm{sgn}(t) + \frac{i}{2}\int_{-\infty}^{\infty}\frac{d\delta(\omega-0)}{d\omega}e^{i\omega t}d\omega\right] \tag{23}$$

and after employing (18), the second term in the rhs of (23) gives:

$$\frac{i}{2}\int_{-\infty}^{\infty}\frac{d\delta(\omega-0)}{d\omega}e^{i\omega t}d\omega=-\frac{i}{2}\int_{-\infty}^{\infty}\delta(\omega-0)ite^{i\omega t}d\omega=\frac{t}{2} \tag{24}$$

Substitution of the result of (24) into (23), gives the causal expression for the impulse response function of the inerter

$$h(t)=\frac{1}{M_R}\left[\frac{t}{2}\mathrm{sgn}(t)+\frac{t}{2}\right]=\frac{1}{M_R}U(t-0)t \tag{25}$$

where $U(t\text{-}0)$ is the Heaviside unit-step function at the time origin(Papoulis 1986, Bracewell 1986). Equation (25) indicates that an impulse force on the inerter creates a causal response that grows linearly with time and is inverse proportional to the inertance, M_R.

The impedance of the inerter as defined by (10) derives directly from (16) by using that $v(\omega)=i\omega u(\omega)$,

$$Z(\omega)=i\omega M_R \tag{26}$$

and is an improper transfer function. Accordingly, its inverse Fourier transform, that is the relaxation stiffness $k(t)$, as defined by (12) does not exist in the classical sense. Nevertheless, it can be constructed mathematically with the calculus of generalized functions and more specifically with the property of the derivative of the Dirac delta function given by (18). By employing (18), the Fourier transform of $d\delta(t\text{-}0)/dt$ is

$$\int_{-\infty}^{\infty}\frac{d\delta(t-0)}{dt}e^{-i\omega t}dt=-\int_{-\infty}^{\infty}\delta(t-0)(-i\omega)e^{-i\omega t}dt=i\omega \tag{27}$$

Consequently, based on the outcome of (27), the inverse Fourier transform of the impedance of the inerter given by (26) is

$$k(t)=M_R\frac{d\delta(t-0)}{dt} \tag{28}$$

Equation (28) indicates that the relaxation stiffness of the inerter exhibits a strong singularity at the time origin given that it is not physically realizable to impose a step displacement to an inerter with finite inertance, M_R.

The admittance (mobility) of the inerter is the inverse of its impedance given by (26):

$$Y(\omega)=\frac{1}{M_R i\omega}=-\frac{1}{M_R}i\frac{1}{\omega} \tag{29}$$

Whereas the admittance (mobility) of the inerter as expressed by (29) is a proper transfer function, the inverse Fourier transform of $-i/\omega$ is $(1/2)\mathrm{sgn}(t)$ (Morse and Feshbach 1953); where, $\mathrm{sgn}(t)$, is the signum function which is clearly a non-causal function. By following the same reasoning described to construct the correct dynamic compliance of the inerter given by (21) we are in search of the real Hilbert pair of the reciprocal function $-1/\omega$ which is $\pi\delta(\omega\text{-}0)$, (Papoulis 1987, Bracewell 1986, Makris 1997a, b and Makris and Kampas 2009). Accordingly, by appending a Dirac delta function as the real part in (29), the correct expression of the admittance of the inerter is

$$Y(\omega)=\frac{1}{M_R}\left[\pi\delta(\omega-0)-i\frac{1}{\omega}\right] \tag{30}$$

and the inverse Fourier transform of the correct admittance of the inerter given by (30) yields

$$y(t)=\frac{1}{M_R}\left[\frac{1}{2}+\frac{1}{2}\operatorname{sgn}(t)\right]=\frac{1}{M_R}U(t-0) \tag{31}$$

which is a causal function since $U(t\text{-}0)$ is the Heaviside unit-step function at the time origin.

5. FREQUENCY- AND TIME-RESPONSE FUNCTIONS OF THE THREE-PARAMETER INERTOVISCOELASTIC FLUID A

The Fourier transform of the constitutive equation of the inertoviscoelastic fluid A given by (2) gives

$$(\omega_R^2+i\omega\lambda\omega_R^2-\omega^2)F(\omega)=-\omega^2(k+i\omega C)u(\omega) \tag{33}$$

Its dynamic compliance, $H(\omega)$, as defined by (6) is

$$H(\omega)=\frac{u(\omega)}{F(\omega)}=-\frac{\omega_R^2+i\omega\lambda\omega_R^2-\omega^2}{\omega^2 k(1+i\omega\lambda)} \tag{33}$$

where $\lambda=C/k$ is the relaxation time and $\omega_R=\sqrt{k/M_R}$ is the rotational frequency of the network. Equation (33) indicates that the dynamic compliance of the inertoviscoelastic fluid A has a double pole at $\omega=0$ and a single pole at $\omega=ik/C=i\lambda$. Partial fraction expansion of the rhs of (33) gives

$$H(\omega)=-\frac{1}{M_R}\frac{1}{\omega^2}-\frac{i}{C(\omega-i/\lambda)} \tag{34}$$

The first term in the rhs of (34) is the dynamic compliance of the solitary inerter as expressed by (17); while, the second term is the dynamic compliance of the Kelvin-Voigt model (a spring and a dashpot connected in parallel). Accordingly, the quadratic singularity,$-1/\omega^2$, that is associated with the dynamic compliance of the solitary inerter is enhanced with its imaginary Hilbert companion as shown by (19) and (20), and the correct expression for the dynamic compliance of the inertoviscoelastic fluid A is

$$H(\omega)=\frac{1}{M_R}\left[-\frac{1}{\omega^2}+i\pi\frac{d\delta(\omega-0)}{d\omega}-\frac{i}{C(\omega-i/\lambda)}\right] \tag{35}$$

Consequently, the dynamic compliance of the inertoviscoelastic fluid A is the superposition of the compliance of the solitary inerter given by (21) and the compliance of the Kelvin-Voigt model (Harris and Crede 1976). The inverse Fourier transform of the dynamic compliance as expressed by (35) gives the causal impulse response function of the inertoviscoelastic fluid A,

$$h(t)=\frac{1}{M_R}U(t-0)t+\frac{1}{C}e^{-t/\lambda} \tag{36}$$

where $U(t\text{-}0)$ is again the Heaviside unit-step function at the time origin.
In the limiting case of a very soft spring ($k\to 0$), the relaxation time $\lambda=C/k$ tends to infinity; and therefore, for positive times ($t\geq 0$),

$$\lim_{\lambda\to\infty} e^{-t/\lambda}=U(t-0) \tag{37}$$

Consequently,

$$\lim_{k\to 0} h(t)=\frac{1}{C}\left[1+\frac{C}{M_R}t\right]U(t-0) \tag{38}$$

which is the impulse response function of a dashpot and an inerter connected in series (Makris 2017).
The impedance of the inertoviscoelastic fluid A derives directly from (32) by using that $v(\omega)=i\omega u(\omega)$ and is given by

$$Z(\omega)=\frac{F(\omega)}{v(\omega)}=\frac{i\omega k-\omega^2 C}{\omega_R^2+i\omega\lambda\omega_R^2-\omega^2} \tag{39}$$

The impedance function given by (39) is a simple proper transfer function, reaching the constant value, *C*, at the high-frequency limit. By separating the high-frequency limit, *C*, the impedance of the mechanical network shown in Fig. 2 (top) is expressed as

$$Z(\omega)=C\left[1+\frac{-\omega_R^2+i\dfrac{\omega}{\lambda}(1-\lambda^2\omega_R^2)}{\omega_R^2+i\omega\lambda\omega_R^2-\omega^2}\right] \tag{40}$$

where the frequency-dependent term in the rhs of (40) is a strictly proper transfer function. The relaxation stiffness, *k*(*t*), of the inertoelastic fluid A is the inverse Fourier transform of the impedance given by (40):

$$k(t)=C\delta(t-0)+\frac{C}{2\pi}\int_{-\infty}^{\infty}\frac{\omega_R^2-i\dfrac{\omega}{\lambda}(1-\lambda^2\omega_R^2)}{(\omega-\omega_1)(\omega-\omega_2)}e^{i\omega t}d\omega \tag{41}$$

where ω_1, ω_2 are the poles of the rhs of (40):

$$\omega_1=\omega_R\sqrt{1-(\frac{\lambda\omega_R}{2})^2}+i\frac{\lambda}{2}\omega_R^2=p+iq \tag{42a}$$

$$\omega_2=-\omega_R\sqrt{1-(\frac{\lambda\omega_R}{2})^2}+i\frac{\lambda}{2}\omega_R^2=-p+iq \tag{42b}$$

The inverse Fourier transform of the rhs of (41) is evaluated with the method of residues and the relaxation stiffness of the three-parameter mechanical network shown in Fig. 2 (top) is

$$k(t)=C\delta(t-0)+\frac{1}{p}\left[\left(k-\frac{C^2}{M_R}\right)\big(p\cos(pt)-q\sin(pt)\big)-\frac{Ck}{M_R}\sin(pt)\right]e^{-qt} \tag{43}$$

where $p=\omega_R\sqrt{1-(\lambda\omega_R/2)^2}$ and $q=\lambda\omega_R^2/2$. Alternatively, by using that $\lambda=C/k$ and $\omega_R^2=k/M_R$ (43) is expressed as

$$k(t)=k\{\lambda\delta(t-0)+[(1-\lambda^2\omega_R^2)(\cos(pt)-\frac{\lambda\omega_R}{2\sqrt{1-(\frac{\lambda\omega_R}{2})^2}}\sin(pt))-\frac{\lambda\omega_R}{\sqrt{1-(\frac{\lambda\omega_R}{2})^2}}\sin(pt)]e^{-qt}\} \tag{44}$$

In the limiting case where the dashpot in the inertoviscoelastic fluid A vanishes, $C=\lambda=q=0$, then $p=\omega_R$ and e^{-qt} tends to $U(t-0)$ for positive times (see (37)). In this limiting case, (43) reduces to

$$\lim_{C\to 0}k(t)=kU(t-0)\cos(\omega_R t) \tag{45}$$

which is the relaxation stiffness of a spring and an inerter connected in series. Alternatively, when the spring in the inertoviscoelastic fluid A vanishes, $k=1/\lambda=\omega_R=0$, then $p=(i/2)C/M_R$ and $q=(1/2)C/M_R$ and (44) reduces to

$$\lim_{k\to 0} k(t) = C\left[\delta(t-0) - \frac{C}{M_R} e^{-\frac{C}{M_R}t}\right] \tag{46}$$

which is the relaxation stiffness of a dashpot and an inerter connected in series (Makris 2017).
When the dimensionless quantity, $\lambda\omega_R$=2, then p=0, q=ω_R and the network shown in Fig. 2 (B) becomes critically damped. In this case ($\lambda\omega_R$=2), (44) assumes the expression

$$\lim_{\lambda\omega_R\to 2} k(t) = k\left[\lambda\delta(t-0) + (-3+\omega_R t)e^{-\omega_R t}\right] \tag{47}$$

Fig. 4 plots the time history of the non-singular component of the normalized relaxation stiffness, $\frac{k(t)}{k} - \lambda\delta(t-0)$, of the inertoviscoelastic fluid A for four values of $\lambda\omega_R$=0.5, 1, 1.5 and 2.
The admittance (mobility) of the inertoviscoelastic fluid A shown in Fig. 2 (A) is the inverse of its impedance as expressed by (39); therefore, it is also a simple proper transfer function. By separating its high-frequency limit, $1/C$, the admittance is expressed as

$$Y(\omega) = \frac{v(\omega)}{F(\omega)} = \frac{1}{C} + \frac{i\omega(\lambda\omega_R^2 - 1/\lambda) + \omega_R^2}{\omega k(i-\omega\lambda)} \tag{48}$$

where the frequency-dependent term in the rhs of (48) is a strictly proper transfer function which has a pole at ω=0 and at ω=i/λ. Accordingly, partial fraction expansion of the frequency-dependent term gives

$$Y(\omega) = \frac{1}{C} - \frac{1}{M_R} i\frac{1}{\omega} + \frac{i}{\lambda C(\omega - i/\lambda)} \tag{49}$$

The second term in the rhs of (49) is the admittance of the solitary inerter as expressed by (29). Accordingly, the singularity, $-i/\omega$, that is associated with the admittance of the solitary inerter is enhanced with its real Hilbert companion, $\pi\delta(\omega$-0), as indicated by (30), and the correct expression for the admittance of the inertoviscoelastic model A is

$$Y(\omega) = \frac{1}{M_R}\left[\pi\delta(\omega-0) - i\frac{1}{\omega}\right] + \frac{1}{C}\left[1 + \frac{i}{\lambda(\omega - i/\lambda)}\right] \tag{50}$$

The second bracket in the rhs of (50) represent the admittance of the Kelvin-Voight model (Harris and Crede 1976); therefore, the admittance of the inertoviscoelastic fluid A is the superposition of the admittance of the solitary inerter given by (30) and the admittance of the Kelvin-Voight model (spring and dashpot in parallel). The inverse Fourier transform of the admittance as expressed by (50) gives the causal impulse velocity response function of the inertoviscoelastic fluid A

$$y(t) = \frac{1}{M_R} U(t-0) + \frac{1}{C}\left[\delta(t-0) - \frac{1}{\lambda} e^{-t/\lambda}\right] \tag{51}$$

where $U(t$-0) is again the Heaviside unit-step function at the time origin. In the limiting case where the spring in the inertoviscoelastic fluid A vanishes, k=1/λ=ω_R=0, (51) reduces to

$$\lim_{k\to 0} y(t) = \frac{1}{C}\delta(t-0) + \frac{1}{M_R} U(t-0) \tag{52}$$

The six basic response functions of the inertoviscoelastic fluid A shown in Figure 2 (A) are summarized in Table I next to the basic response functions of the solitary inerter.
The basic time response functions of the other fluid-like models are extracted after applying similar

mathematical techniques and are shown in Table1.

8. CONCLUSIONS

This paper derives the causal time-response functions of three-parameter mechanical networks which involve the inerter—a two-node element in which the force-output is proportional to the relative acceleration of its end-nodes. This is achieved by extending the relation between the causality of a time-response function with the analyticity of its corresponding frequency response function to the case of generalized functions.

The integral representation of the output signals presented in this study offers an attractive computational alternative given that the constitutive equations of some of the three-parameter models examined involve the third derivative of the nodal displacement (derivative of the acceleration) which may challenge the numerical accuracy of a state-space formulation given that in several occasions the input signal is only available in digital form as in the case of recorded accelerograms.

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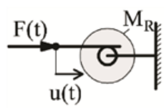
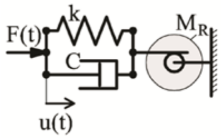
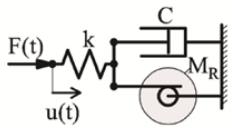
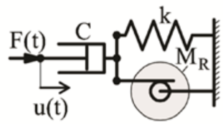
Table 1. Basic Frequency-response Functions and Their Corresponding Causal Time-response Functions of the Three-parameter Inertoviscoelastic Fluid A, B And C.

$\lambda=\frac{C}{k}$ $\omega_R^2=\frac{k}{M_R}$ $p=\omega_R\sqrt{1-\left(\frac{\lambda\omega_R}{2}\right)^2}$ $q=\frac{1}{2}\lambda\omega_R^2$	Inerter	Inertoviscoelastic Fluid A	Inertoviscoelastic Fluid B	Inertoviscoelastic Fluid C
				
Constitutive Equation	$F(t)=M_R\frac{d^2u(t)}{dt^2}$	$F(t)+\lambda\frac{dF(t)}{dt}+\frac{1}{\omega_R^2}\frac{d^2F(t)}{dt^2}=M_R\left(\frac{d^2u(t)}{dt^2}+\lambda\frac{d^3u(t)}{dt^3}\right)$	$F(t)+\lambda\frac{dF(t)}{dt}+\frac{1}{\omega_R^2}\frac{d^2F(t)}{dt^2}=C\left(\frac{du(t)}{dt}+\frac{M_R}{C}\frac{d^2u(t)}{dt^2}\right)$	$F(t)+\lambda\frac{dF(t)}{dt}+\frac{1}{\omega_R^2}\frac{d^2F(t)}{dt^2}=C\left(\frac{du(t)}{dt}+\frac{1}{\omega_R^2}\frac{d^3u(t)}{dt^3}\right)$
Dynamic Compliance or Flexibility $H(\omega)=\frac{u(\omega)}{F(\omega)}$	$\frac{1}{M_R}[-\frac{1}{\omega^2}+i\pi\frac{d\delta(\omega-0)}{d\omega}]$	$\frac{1}{M_R}[-\frac{1}{\omega^2}+i\pi\frac{d\delta(\omega-0)}{d\omega}-\frac{i}{C(\omega-i/\lambda)}]$	$\frac{1}{k}[1+\frac{1}{\lambda}(\pi\delta(\omega-0)-i\frac{1}{\omega}+\frac{i}{\omega-i\lambda\omega_R^2})]$	$\frac{1}{C}[\pi\delta(\omega-0)-i\frac{1}{\omega}]+\frac{1}{2M_R\omega_R}[\frac{1}{\omega+\omega_R}-\frac{1}{\omega-\omega_R}+i\pi(\delta(\omega+\omega_R)-\delta(\omega-\omega_R))]$
Impedance $Z(\omega)=\frac{F(\omega)}{v(\omega)}$	$0+i\omega M_R$	$C\left[1+\frac{-\omega_R^2+i\frac{\omega}{\lambda}(1-\lambda^2\omega_R^2)}{\omega_R^2+i\omega\lambda\omega_R^2-\omega^2}\right]$	$k\frac{\lambda\omega_R^2+i\omega}{\omega_R^2+i\omega\lambda\omega_R^2-\omega^2}$	$C\left[1-\frac{i\omega\lambda\omega_R^2}{\omega_R^2+i\omega\lambda\omega_R^2-\omega^2}\right]$
Admittance or Mobility $Y(\omega)=\frac{v(\omega)}{F(\omega)}$	$\frac{1}{M_R}[\pi\delta(\omega-0)-i\frac{1}{\omega}]$	$\frac{1}{M_R}[\pi\delta(\omega-0)-i\frac{1}{C}]+\frac{1}{C}[1+\frac{i}{\lambda(\omega-i/\lambda)}]$	$\frac{1}{k}\left(i\omega+\frac{\omega_R^2}{\lambda\omega_R^2+i\omega}\right)$	$\frac{1}{C}+\frac{1}{2M_R}[\pi\delta(\omega-\omega_R)-i\frac{1}{\omega-\omega_R}+\pi\delta(\omega+\omega_R)-i\frac{1}{\omega+\omega_R}]$
Impulse Response Function $h(t)$	$\frac{1}{M_R}U(t-0)t$	$\frac{1}{M_R}U(t-0)t+\frac{1}{C}e^{-t/\lambda}$	$\frac{1}{k}\left[\delta(t-0)+\frac{1}{\lambda}[U(t-0)-e^{-\frac{C}{M_R}t}]\right]$	$[\frac{1}{C}+\frac{1}{M_R\omega_R}\sin(\omega_R t)]U(t-0)$
Relaxation Stiffness $k(t)$	$M_R\frac{d\delta(t-0)}{dt}$	$k\{\lambda\delta(t-0)+[(1-\lambda^2\omega_R^2)(\cos(pt)-\frac{\lambda\omega_R}{2\sqrt{1-(\frac{\lambda\omega_R}{2})^2}}\sin(pt))-\frac{\lambda\omega_R}{\sqrt{1-(\frac{\lambda\omega_R}{2})^2}}\sin(pt)]e^{-qt}\}$	$k\left(\cos(pt)+\frac{\lambda\omega_R}{\sqrt{4-\lambda^2\omega_R^2}}\sin(pt)\right)e^{-qt}$	$k[\lambda\delta(t-0)-\lambda^2\omega_R^2(\cos(pt)-\frac{\lambda\omega_R}{\sqrt{4-\lambda^2\omega_R^2}}\sin(pt))e^{-qt}]$
Impulse Velocity Response Function $y(t)$	$\frac{1}{M_R}U(t-0)$	$\frac{1}{M_R}U(t-0)+\frac{1}{C}[\delta(t-0)-\frac{1}{\lambda}e^{-t/\lambda}]$	$\frac{1}{k}\frac{d\delta(t-0)}{dt}+\frac{1}{M_R}e^{-\frac{C}{M_R}t}$	$\frac{1}{C}\delta(t-0)+\frac{1}{M_R}U(t-0)\cos(\omega_R t)$

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