EXAMINING THE BASIS FOR VISCOUS DAMPING IN SEISMIC ANALYSIS

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ABSTRACT

Viscous damping is used as simple mathematical model to introduce energy dissipation in structures during dynamic excitation. Viscous damping preserves the linearity of the governing differential equations and, in the case of proportional damping, the orthogonal eigenmodes of the undamped structure. For those reasons, viscous damping is a convenient model, and it is used habitually in seismic analysis and other types of dynamic analysis. However, a fundamental problem rarely addressed is the correct interpretation of the viscous forces that are present in the model. This paper identifies two conflicting interpretations: one is called Hookean and the other is called Newtonian. The paper shows how each interpretation is valid in various circumstances and discusses the implications.

Rayleigh damping is by far the most popular damping model for analysis of multi degrees-of-freedom systems in time domain. The use of Rayleigh damping is associated with some problems that are not widely recognised, especially in the context of seismic analysis. The paper reviews key findings which show that Rayleigh damping violates a fundamental law of physics and can lead to incompatible results, depending on the adopted frame of reference. A generic model with two degrees-of-freedom is examined. The results indicate that the use of Rayleigh damping can be problematic for structures responding in the displacement-sensitive range of the input spectrum.

Keywords: Rayleigh damping; Proportional damping.

1. INTRODUCTION

The equations of motion governing the response of vibrating systems are simple to derive and solve for linear-elastic systems. The real challenge, in many cases, is to account for inelastic effects causing energy dissipation and hence damping. Characterisation of energy dissipation in a vibrating system has been an active area of research for more than a century, but a single unifying framework, founded in physical laws, has proved elusive. In buildings and other civil engineering structures, energy dissipation occurs due to friction in joints and other interfaces, hysteretic material behaviour, radiation damping, interaction with non-structural components, etc. Each effect requires relatively complex modelling for proper characterisation. However, for numerical analysis, it is desirable and sometimes necessary to represent these effects by means of simpler models.

To account for energy dissipation in vibrating systems, the simplest and most widely used model is viscous damping. However, the use of viscous damping is not without problems. The purpose of this paper is to identify and discuss some of those problems on a fundamental level. To this end, the paper provides a survey of the basic theory of viscous damping. The paper discusses the interpretation of the viscous forces that are present in a model. The paper highlights some fundamental concerns with Rayleigh damping, which is the most popular type of viscous damping, and provides an example to illustrate the importance of the problems for a generic two degrees-of-freedom (DOF) system with a low level of damping.

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2. SURVEY OF BASIC THEORY

2.1 Single degree-of-freedom system

The equation of motion for a linear-elastic single degree-of-freedom (SDOF) system with viscous damping may be written as:

\[ m\ddot{w} + c\dot{w} + kw = f_E \]  

(1)

where \( m \) is the mass, \( c \) is the damping coefficient, \( k \) is the stiffness coefficient, \( f_E \) is an external force and \( w \) is the displacement. A dot denotes differentiation with respect to time \( t \).

Equation 1 may also be written as:

\[ f_I + f_D + f_S = f_E \]  

(2)

where \( f_I \) is the inertial force, \( f_D \) is the damping force and \( f_S \) is the spring force (or elastic force). The \text{internal or resistive} forces in this system are defined as:

\[ f_R = f_D + f_S \]  

(3)

In the case of earthquake excitation defined by base motion \( s \), the external force is given by:

\[ f_E = -m\ddot{s} \]  

(4)

In the remainder of this paper only cases of earthquake excitation are considered. Other types of loading are not considered. This is not, however, a significant limitation on the relevance of this paper.

2.2 Multi degree-of-freedom system

For a multi degrees-of-freedom (MDOF) system, the equations of motion can be written as

\[ [M]\{\ddot{w}\} + [C]\{\dot{w}\} + [K]\{w\} = -[M]\{\ddot{s}\} \]  

(5)

where \([M]\), \([C]\) and \([K]\) are the mass, damping and stiffness matrices, and \( \{w\} \) denotes a vector of displacements. The vector \( \{s\} \) is defined as an expansion of the fundamental components of the base motion \( \{s_x, s_y, s_z\} \) to all DOF. Rotational components of the base motion are assumed equal to zero. In general, the system has \( N \) DOF, and each matrix is an \( N\times N \) square and symmetric matrix.

It is necessary to distinguish between base motion \( \{s\} \), total displacements \( \{u\} \) and relative displacements \( \{w\} \). The three quantities are related by:

\[ \{w\} = \{u\} - \{s\} \]  

(6)

Equations 1 and 5 apply in an accelerated frame of reference that follows the base motion. The equations of motion in an inertial frame of reference are given by:

\[ [M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{0\} \]  

(7)

The equations of motion may be partitioned into free DOF (denoted by the subscript \( a \)) and prescribed DOF at the base of the structure (denoted by the subscript \( b \)). Writing Equation 5 in partitioned format gives:
\[
\begin{bmatrix}
M_{aa} & M_{ab} \\
M_{ba} & M_{bb}
\end{bmatrix}
\begin{bmatrix}
\ddot{w}_a \\
\ddot{w}_b
\end{bmatrix}
+ \begin{bmatrix}
C_{aa} & C_{ab} \\
C_{ba} & C_{bb}
\end{bmatrix}
\begin{bmatrix}
\dot{w}_a \\
\dot{w}_b
\end{bmatrix}
+ \begin{bmatrix}
K_{aa} & K_{ab} \\
K_{ba} & K_{bb}
\end{bmatrix}
\begin{bmatrix}
w_a \\
w_b
\end{bmatrix}
= -\begin{bmatrix}
M_{aa} & M_{ab} \\
M_{ba} & M_{bb}
\end{bmatrix}
\begin{bmatrix}
\ddot{s}_a \\
\ddot{s}_b
\end{bmatrix}
\tag{8}
\]

Imposing boundary conditions \(\{w_b\} = \{0\}\) gives:

\[
[M_{aa}] \{\ddot{w}_a\} + [C_{aa}] \{\dot{w}_a\} + [K_{aa}] \{w_a\} = -[M_{aa}] \{\ddot{s}_a\} - [M_{ab}] \{\ddot{s}_b\}
\tag{9}
\]

In the remainder of this paper, all matrix equations are based on the submatrices \([M_{aa}], [C_{aa}] \& [K_{aa}]\). Subscript \(aa\) is omitted, but implied everywhere unless otherwise noted.

### 2.3 Viscous damping parameters

The theory of SDOF and MDOF systems with viscous damping is well-known (see for example Clough & Penzien, 1995, for a comprehensive treatment), and only the salient results are repeated here, without derivation.

The damping ratio \(\xi\) for an SDOF system with viscous damping is defined as:

\[
\xi = \frac{c}{2m\omega_n}
\tag{10}
\]

where \(\omega_n\) is the natural frequency of the undamped system:

\[
\omega_n = \sqrt{\frac{k}{m}}
\tag{11}
\]

Most measurements of damping are performed under conditions of free vibration or steady-state forced vibration at, or near, resonance. Although the experimental evidence in general does not conform to the viscous damping model, it is common practice to evaluate an equivalent viscous damping ratio \(\xi_{eq}\), which ensures that Equation 1 gives the right magnitude of response at resonance. Consider a steady-state harmonic test conducted at resonance. The excitation is applied as base motion with sinusoidal variation:

\[
s(t) = s_0 \sin(\omega_f t)
\tag{12}
\]

where \(\omega_f\) is the excitation frequency. The test involves establishing resonance by adjusting the excitation frequency until the displacement is 90° out-of-phase with the applied force. At this phase angle, the viscous damping force would be equal and opposite to the applied force. The area enclosed within the hysteresis loop, \(E_D\), on a plot of internal forces vs. displacement is equal to the energy dissipated per cycle (Figure 1). Equating this area to the energy dissipated by a viscous damper gives the equivalent damping ratio as:

\[
\xi_{eq} = \frac{E_D}{2\pi \sqrt{km\omega_f} w_0^2}
\tag{13}
\]

where \(w_0\) is the amplitude of the steady-state response. Bert (1973) provides several closed-form expressions for \(\xi_{eq}\) corresponding to various non-viscous damping models.
2.4 Proportional damping models

The viscous model is convenient because it preserves the linearity of the governing differential equations. A certain type of viscous damping known as ‘classical damping’ or ‘proportional damping’ is popular for many reasons including: (1) it preserves the orthogonality of the undamped eigenmodes; (2) it gives unique control over damping ratios for each mode; (3) there is a general lack of alternative models to choose from (noteworthy exceptions to this last point are the hysteretic material damping implemented in FLAC3D and the so-called structural damping based on an imaginary stiffness).

The most common type of proportional damping is Rayleigh damping, in which the damping matrix is evaluated as a linear combination of the mass and stiffness matrices:

\[
[C] = a_0[M] + a_1[K] \tag{14}
\]

Rayleigh damping is a special case of a more general formula:

\[
[C] = [M] \sum_{b \in B} a_b[M^{-1}K]^b \tag{15}
\]

where \(B\) is a set of integers. Equation 15 was first proposed by Caughey (1960), albeit in a slightly different form, and is therefore often called a Caughey series. Rayleigh damping is obtained when \(B = \{0,1\}\). If the first \(p\) modal damping ratios are prescribed, such that \(p \leq N\) and \(B = \{0, 1, 2, \ldots, p-1\}\), then a unique solution for the \(p\) unknown constants \(a_b\) in Equation 15 is given by (e.g., Luco, 2007):

\[
\begin{bmatrix}
1 & \omega_1^2 & \omega_1^4 & \cdots & \omega_1^{2p-2} \\
1 & \omega_2^2 & \omega_2^4 & \cdots & \omega_2^{2p-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega_p^2 & \omega_p^4 & \cdots & \omega_p^{2p-2}
\end{bmatrix} \begin{bmatrix}
a_0 \\
a_1 \\
\vdots \\
a_{p-1}
\end{bmatrix} = \begin{bmatrix}
2\omega_1\xi_1 \\
2\omega_2\xi_2 \\
\vdots \\
2\omega_p\xi_p
\end{bmatrix} \tag{16}
\]

where \(\omega_n\) and \(\xi_n\) are the natural frequency and damping ratio, respectively, for mode \(n\). The square matrix in Equation 16 is a Vandermonde matrix. Unfortunately, the Vandermonde matrix becomes increasingly ill-conditioned as \(p\) increases and/or when \(\omega_p \gg \omega_1\). Luco’s proposal (2007) for an alternative computation of \([C]\) based on Equation 15 avoids the inversion of the Vandermonde matrix, but is still computationally intensive. Another concern associated with the solution provided by Equation 16 is that...
the damping ratios for modes \( p+1 \) to \( N \) are not controlled and could even be negative.

Wilson and Penzien (1972) derived another method based on an inverse modal transformation method, in which the damping matrix is computed directly from the mode shapes. In this method, the first \( p \) natural modes are extracted and the damping matrix is obtained by:

\[
[C] = 2 \xi_p \frac{[K]}{\omega_p} + 2[\tilde{M}] \left[ \sum_{n=1}^{p-1} \left( \xi_n - \frac{\xi_n}{\omega_p} \right) \phi_n^T \phi_n \right] [\tilde{M}]
\]  

(17)

where \( \{ \phi_n \} \) is the mode shape of mode \( n \) (normalised with respect to the mass matrix). Equation 17 is more convenient to apply than Equation 16 and offers complete control over all damping ratios. The purpose of the first, stiffness-proportional term in Equation 17 is to ensure that modes above the \( p \)th mode are damped. In this formulation, the damping ratios for the higher modes increase in proportion to frequency, but this is often acceptable since the higher modes contribute little to the total response.

Adhikari (2006) showed that the Caughey series is a special case of an even more general type of proportional damping. He proved that a viscously damped linear system will have proportional damping if and only if the damping matrix can be represented by one of the following equations:

\[
(a) \quad [C] = [\tilde{M}] \beta_1 \left( [M^{-1}] \tilde{K} \right) + [K] \beta_2 \left( [M^{-1}] \tilde{K} \right) \\
(b) \quad [C] = \beta_3 \left( [M^{-1}] \tilde{K} \right) [\tilde{M}] + \beta_4 \left( [M^{-1}] \tilde{K} \right) [K]
\]

(18)

where \( \beta(\bullet) \) are smooth analytic functions in the neighbourhood of all of the eigenvalues of their argument matrices. Adhikari went on to show that the damping matrix can be obtained by fitting a function to specified or measured data points of modal damping ratios. However, the method developed by Adhikari still involves calculating the square root of \([M^{-1} K]\), which in practice requires computation of all eigenvalues and eigenmodes. As such, Adhikari’s method offers little benefit over the Wilson-Penzien method.

Recall that the subscript \( aa \) is implied on all matrices (see paragraph after Equation 9): it follows that Equations 14-18 are all dependent on the boundary conditions. This illustrates why the proportional damping model should not be understood as a material property, but perhaps rather as an advanced boundary condition (Lindsay, 1929), much in the same manner as radiation damping is modelled by means of advanced boundary conditions.

3. THE INTERPRETATION PROBLEM

3.1 Problem statement and discussion

Viscous damping was introduced as a convenient model that can replicate several manifestations of energy dissipation observed in real situations such as decay of free vibration. The model is remarkably successful in this regard; we know that the model is a mathematical abstraction in most cases, but it enables us to predict the response of many real systems with satisfactory accuracy. The viscously damped SDOF system comprises two internal forces, one that is conservative (the spring force \( f_S \)) and one that is non-conservative (the damping force \( f_D \)). The internal forces are in dynamic equilibrium with the externally applied force \( f_E \) and the inertial force \( f_I \) according to d’Alembert’s principle. These basic considerations, which can be extended to MDOF systems, are well understood and accepted.

For many engineering applications, we are interested in the peak force within the spring (the peak force being the maximum absolute value). The spring may represent a real spring, a single structural member such as a bar or a beam, or even a structure with multiple members. The peak spring force is evaluated
from the displacement time history $w(t)$ as:

$$f_{Sp} = k \max |w|$$

(19)

In general, the velocity is always zero at the instant of peak displacement, and so if $f_{Sp}$ occurs at time $t$, then $f_D = 0$ at the same time, and therefore $f_{Sp}$ is equal to the absolute sum of $f_E$ and $f_I$ at that instant in time. However, the peak sum of $f_E$ and $f_I$ is not necessarily equal to $f_{Sp}$. In fact, the peak internal force required to balance the peak sum of $f_E$ and $f_I$ is given by:

$$f_{Rp} = \max |f_S + f_D| = \max |kw + cv|$$

(20)

It is not immediately clear whether we ought to use $f_{Sp}$ or $f_{Rp}$ for assessment or design of the spring. Two interpretations are possible. On one hand, we acknowledge that the viscous force is a mathematical abstraction, and that the force in the spring must satisfy Hooke’s Law, and on this basis we conclude that the peak force in the spring is equal to $f_{Sp}$ (this may be termed a Hookean interpretation – *ut tensio sic vis*). On the other hand, we acknowledge that to satisfy the principle of dynamic equilibrium at the critical instant in time, to satisfy Newton’s Second Law, we have to provide a structural element capable of sustaining a dynamic force equal to $f_{Rp}$ (this may be termed a Newtonian interpretation – *mutationem motus proportionalem esse vi motrici impressae*). Which one is correct?

This question represents a fundamental interpretation problem associated with viscous damping – a problem that has not received much attention within the literature to date. In the author’s experience, the default interpretation is the Hookean. For example, in using the spectral pseudo-acceleration for seismic design of structures, which is the norm, we have implicitly opted for a Hookean interpretation. If instead we were to use the ‘true’ spectral acceleration, then we would have opted for a Newtonian interpretation. The spectral pseudo-acceleration is approximately equal to the true spectral acceleration for low levels of damping and over a broad range of frequencies; the difference between the two accelerations becomes significant only at frequencies below 0.1 Hz (Chopra, 2001). Therefore, for many linear-elastic systems with low levels of damping, the difference between the two models is not significant, and presumably we can use one or the other without having to justify our choice.

In his comprehensive textbook, Chopra (2001) aligns with the Hookean interpretation. He notes that it is “inappropriate to include the velocity-dependent damping force because for structural design the computed element stresses are to be compared with allowable stresses that are specified based on static tests on materials (i.e. tests conducted at slow loading rates).” This justification appears tenuous. It is well known that many materials exhibit higher strength under high strain rates; however, such increases in strength are normally only observed under strain rates produced by blast and impact loading and occur on timescales of milli-seconds. For ‘slower’ types of dynamic loading, such as earthquake excitation, the damping forces may be significant even when strain rates are not.

Although the Hookean interpretation is far more common, there are cases where a Newtonian interpretation has been adopted (perhaps unintentionally so). In ANSYS, for example, the reaction forces at the base of the structure are calculated as (cf. Equation 8):

$$\{F_{Rh}\} = [M_{ab}]\{\ddot{u}_b\} + [C_{ab}]\{\dot{w}_b\} + [K_{ab}]\{w_b\}$$

(21)

The reaction forces calculated in this manner achieve dynamic equilibrium. However, within a Hookean framework, the internal forces in the elements that connect to the base should be calculated as:

$$\{F_{Sh}\} = [K_{ab}]\{w_b\}$$

(22)

The difference between Equations 21 and 22 highlights an inconsistency in the approach as the forces used for design of the foundation may be greater than the design forces used for design of the first storey.
columns or walls. As shown below, this problem may be aggravated if Rayleigh damping is used and the mass-proportional term in Equation 14 is taken into account.

A serious problem already investigated by several researchers arises when a viscous damping model is used in combination with inelastic material response (e.g. Priestley & Grant, 2005; Hall, 2006; Jehel et al., 2014). It is beyond the scope of this paper to discuss the problems in depth; however, one example may provide an appreciation of the difficulties. Bernal (1994) showed that systems with massless or nearly massless DOF can develop abnormally high damping forces upon the onset of yielding or other inelastic effects characterised by sudden changes in stiffness. The changes in stiffness cause sudden changes in velocity and hence abnormally high damping forces. Observing that the equilibrium at massless DOF should be static in nature, not only for elastic behaviour but in general, Bernal again opted for a Hookean interpretation and concluded that the damping forces computed there are spurious. One may counter that massless DOF are themselves a mathematical abstraction as no solid is without mass. However, the findings suggest that the combination of viscous dampers and inelastic elements in the same model requires particular care. The remedy proposed by Bernal is to remove the damping forces from the massless DOF, either by limiting the summation in the Caughey series to zero or negative b-values, or by performing static condensation where possible prior to the dynamic analysis.

3.2 Example – SDOF system subject to harmonic excitation

Suppose we have a real SDOF system that displays a bilinear response with kinematic hardening. Such a system can be characterised by 4 parameters: mass \( m \), initial stiffness \( k \), yield force \( f_y \), and post-yield stiffness ratio \( r \). The initial natural frequency is given by Equation 11. In this hypothetical example, units are not important. Upon the onset of yielding, the tangent stiffness is given by:

\[
k_t = rk
\]

The ductility ratio \( \mu \) is defined in the usual manner as the ratio of the peak displacement \( w_p \) to elastic displacement \( w_y \) where:

\[
w_y = \frac{f_y}{k}
\]

Our aim is to approximate this real system by means of an equivalent linear-elastic system with viscous damping. We subject the real system to harmonic base excitation at resonance (Equation 12, with \( \omega_f = \omega_n \)) and measure the steady-state response; we then fit the best possible linear model to the measured response. The equivalent linear stiffness is chosen as:

\[
k_e = \frac{f_p}{w_p}
\]

The equivalent damping ratio can be derived from Equation 13 as (cf. Priestley & Grant, 2005):

\[
\zeta_e = \frac{2(\mu - 1)(1 - r)}{\pi \mu (1 + r(\mu - 1))}
\]

Figure 2(b) shows the response at resonance. The test gives \( k_e = 0.82k \) and \( \zeta_e = 7\% \), which is typical for reinforced concrete and friction-bolted structures (note: in this case, the yield force \( f_y \) would not represent complete section yielding, but rather the onset of cracking or sliding within joints). The test shows that the linear model provides a good fit to the real system. Clearly, the total internal force \( f_R \) provides an overall better fit to the internal force in the real system than the spring force \( f_S \), and this observation indicates that a Newtonian interpretation is appropriate. However, the ratio of peak forces \( f_{Sp} / f_{Rp} = 0.99 \),
even when the transient response is considered, so evidently for design purposes it is irrelevant to ask for an interpretation.

Figure 2(a) shows the steady-state response of the real system and the equivalent linear system at a relatively slow rate of excitation. The properties of each system are unchanged. The amplitude of the base motion was selected such that the response amplitude is approximately the same as in Figure 2(b). At this lower excitation frequency, the energy dissipation per unit cycle is significantly lower in the equivalent linear system; however, the linear system still provides a good prediction of the response amplitude, and the peak spring force \( f_{Sp} \) is nearly identical to the peak total force \( f_{Rp} \).

Figure 2(c) shows the steady-state response at a relatively high excitation frequency. Again, the properties of each system are unchanged, and only the amplitude of base motion was adjusted to give the same bilinear system response as in Figure 2(b). Remarkably, the peak displacement of the equivalent system is nearly equal to that of the real system even though the energy dissipation in the equivalent system is much higher. Figure 2(c) also shows that the peak spring force \( f_{Sp} \) is no longer comparable to the peak total force \( f_{Rp} \) at higher frequencies (\( f_{Sp} \)/\( f_{Rp} = 0.85 \)). However, the peak spring force in the equivalent system is still a good approximation of the peak internal force in the real system, so now a Hookean interpretation appears more appropriate.

This example shows that, depending on our interest in the force-displacement graphs, one interpretation may appear more valid than the other. The hysteretic loops are better represented by the total force (Newtonian interpretation), but at relatively high excitation frequencies the total force tends to overshoot, and the peak response, at least, is more closely approximated by the spring force (Hookean interpretation).

Figure 2. Response of bilinear (with \( r = 0.6 \)) and equivalent viscous systems

4. THE PROBLEM WITH RAYLEIGH DAMPING

4.1 General considerations

Rayleigh damping is by far the most popular variant of viscous damping for MDOF structures. For that reason, several researchers have tried to optimise its implementation with varying degrees of success. Others have pointed to various shortcomings or pitfalls. Hall (2006) summarised the problems; he identified that the mass-proportional term represents the effects of a viscous ether, in which the structure is immersed, and that this cannot exist for a real structure. Moreover, the mass-proportional term can lead to unrealistically high damping forces whenever some or all parts of the structure behave like a
rigid body. Nielsen (2009) showed that the mass-proportional term does not remain invariant under a Galilean transformation and is, therefore, physically impossible. Therefore, interpreting the mass-proportional term as a material property is clearly wrong (and likely the same can be said for the stiffness-proportional term); however, both Hall and Nielsen recognised that Rayleigh damping may be used as a heuristic damping model, or perhaps as a boundary condition in a wider sense of the term, as long as we are aware of its limitations. It should be noted that the mass-proportional term is present in all proportional models unless we take specific measures to exclude it from the formulation (e.g. by setting $\beta_3 = 0$ in Equation 18).

The key requirement highlighted by Nielsen (2009) is that the results of an analysis must be independent of the frame of reference chosen for the analysis. In seismic analysis, we have a choice of two frames: an inertial frame, using total displacements $\{u\}$; or an accelerated frame, using relative displacements $\{w\}$. The independence requirement is not satisfied by commercial finite element programs such as ANSYS and Abaqus when mass-proportional damping is used. Crucially, this means that the analyst should not expect to obtain the same results when switching from one frame to another. The only way to achieve complete equivalence between the two solution methods is to assume that the mass-proportional dampers are connected to the foundation. This assumption is implicit in Equation 5 once boundary conditions $\{w_b\} = \{0\}$ have been imposed, but not necessarily in Equation 7. Essentially, the assumption means that the mass-proportional term becomes physically possible, even if it remains extremely difficult to realise in an actual structure.

4.2 Example – 2DOF system subject to earthquake excitation

The presence of mass-proportional dampers in the model complicates the interpretation problem discussed above. To explain this, consider a 2 degrees-of-freedom (2DOF) system as shown in Figure 3. The system contains two subsystems designated primary ($p$) and secondary ($s$) as per convention. The peak force in the secondary spring is:

$$f_{s,Sp} = k_s \max \left| w_p - w_s \right|$$

(27)

When this peak force occurs, the viscous force in the secondary stiffness-proportional dashpot is zero. However, the viscous force in the mass-proportional dashpot is not necessarily zero. Therefore, the mass-proportional dashpot could influence the value of $f_{s,Sp}$ depending on the relative phasing of the response.

On a more fundamental level, the mass-proportional dashpots provide an alternative load path to the foundation, bypassing the primary subsystem. This is not a problem within a Hookean interpretation where the viscous forces are ignored, but it clearly poses a problem within a Newtonian interpretation where the viscous forces are seen as part of the load-resisting system.

A study was undertaken to quantify the difference between the two interpretations. SeismoArtif version 2.1.0 was used to develop eight artificial accelerograms matching the PML spectrum (a UK design
spectrum used for nuclear safety-related structures) for medium ground conditions (Figure 4). The velocity-sensitive frequency range of the PML spectrum depends on the damping level; for 5% damping the range is between 0.9 and 3.7 Hz. The duration of each time history was 14 seconds.

Figure 4. Target PML spectrum at 5% damping and spectra from eight matching accelerograms

The accelerograms were used as input to eight analyses of the 2DOF system shown in Figure 3. The fundamental system parameters are: natural frequency of the primary system $f_p$, mass of primary system $m_p$, and damping ratios for the 1st and 2nd modes, $\xi_1$ and $\xi_2$. Three different values of $f_p$ were considered (0.5, 2 & 6 Hz); the mass of the primary subsystem was given a value of unity; and the damping ratios were chosen as: $\xi_1 = \xi_2 = 7\%$. The other system parameters are defined through the following ratios:

Mass ratio:  
$$ R_m = \frac{m_s}{m_p} $$  

Frequency ratio:  
$$ R_f = \frac{f_s}{f_p} $$

For this simple system, the three advanced versions of proportional damping (Equations 15, 18 & 17) give the same damping matrix as Rayleigh damping (Equation 14).

Initially, the frequency of the primary system was chosen as $f_p = 0.5$ Hz, well into the displacement-sensitive region of the input spectrum. Eight transient analyses were carried out, using the input described above. The results from each analysis were peak dynamic reaction $F_{Rp}$ evaluated as per Equation 21 and peak primary spring force $F_{Sp}$ evaluated as per Equation 22 (taking the maximum absolute value over each time history). The ratio $F_{Sp} / F_{Rp}$ was evaluated, and the mean of all eight analyses was then plotted as a function of $R_f$ for five values of $R_m$. This process was repeated for $f_p = 2$ Hz (velocity-sensitive region) and $f_p = 6$ Hz (acceleration-sensitive region) – see Figure 5.

The results show that the mean force ratio $F_{Sp} / F_{Rp}$ is consistently between 1 and 0.95 for $R_m \leq 1$, regardless of the primary subsystem frequency. Thus, when the secondary subsystem is lighter than the primary subsystem, the difference between the two approaches is insignificant. When the frequency of the primary subsystem is within the displacement-sensitive region, the mean force ratio $F_{Sp} / F_{Rp}$ may be less than 0.6 with high mass ratios – a situation that typifies base-isolated structures. However, when the frequency of the primary subsystem is within the acceleration-sensitive region, the mean force ratio is between 1 and 0.95 for all $R_m \leq 100$. 

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Based on these results, it may be concluded that, except for certain cases where a low-frequency primary subsystem supports a relatively heavy secondary subsystem, the interpretation problem is mainly a theoretical problem with little relevance for practical design situations. However, this conclusion may not be valid for models with high levels of damping (say, $\xi > 10\%$) or for non-linear models with viscous damping; for such models, it remains pertinent to validate the damping forces and to assess any major discrepancies between the spring force $f_S$ and the total force $f_R$.

6. CONCLUDING REMARKS AND RECOMMENDATIONS

This paper examined the basis of viscous damping in seismic analysis. The paper highlighted that the analyst has a choice when he or she evaluates the internal forces in the structural elements: either to exclude the viscous forces or to include them. In the first case, the analyst subscribes to a Hookean interpretation; in the second case, to a Newtonian interpretation. However, unless the structure contains a real viscoelastic material, the viscous damper is largely a mathematical artifice influencing both displacements and accelerations. Therefore, neither interpretation is strictly speaking correct.

In the context of force-based seismic design, it should be verified that the peak elastic force $f_{Sp}$ is sufficiently close to the peak total force $f_R$ (the term ‘force’ should be understood in a general sense as covering not only forces but also moments and stresses). This ensures that internal forces in the elements are consistent with the relative displacements (the elastic forces) and the total accelerations (the inertial forces).
Commercial finite element programs do not contain a consistent implementation of Rayleigh damping. Until this is mended, systems with Rayleigh damping should be analysed in an accelerated frame of reference. It should be noted that the mass-proportional damping forces are probably excluded from the total reactions, even when the software purports to include viscous forces. This is likely a negligible oversight unless the system responds in the displacement-sensitive range of the input spectrum.

The following structural types should receive particular attention:

- Structures responding in the displacement-sensitive range of the input spectrum (e.g. base-isolated structures).
- Structures supporting relatively large masses.
- Structures with rigid or nearly rigid body modes.
- Structures subject to high-frequency excitation (relative to their natural frequencies).
- Structures with yielding elements or other nonlinear effects in combination with viscous damping.

7. ACKNOWLEDGMENTS

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8. REFERENCES


