16TH EUROPEAN CONFERENCE ON EARTHQUAKE ENGINEERING
THESSALONIKI 18 - 21 JUNE 2018

FREE VIBRATION ANALYSIS OF A PLANAR ELLIPTICAL BEAM

Merve ERMIS[1], Umit N ARIBAS[2], Nihal ERATLI[3], Mehmet H OMURTAG[4]

ABSTRACT

The free vibration analysis of a planar elliptical Timoshenko beam with two different orientations of an elliptical cross-section about beam axis is investigated and the results are compared by the circular cross-section keeping the net area for all cases equal to each other. The functional is based on the Gateaux differential and the potential operator concept. The planar elliptical beam is discretized by a two-noded curvilinear mixed finite elements. Each node of the element has 12 DOFs, namely, three translations, three rotations, two shear forces, one axial force, two bending moment and one torque. The finite element matrices are derived by using the exact nodal geometrical parameters (arc length, curvature, and torsion) of the curvilinear element and these parameters are linearly interpolated through the element. Since the solution of the free vibration analysis reduces to an eigenvalue problem, a condensation procedure is applied on the system matrix over the stress resultants. A parametric study is performed to investigate the influence of the orientation of the elliptical cross sectional geometry on the natural frequencies of the planar elliptical beam.

Keywords: elliptical beam; Timoshenko beam theory; free vibration analysis; finite element method

1. INTRODUCTION

The curved beam elements are increasingly used in mechanical, civil, mechatronics and aerospace engineering such as bridges, railways, aircrafts, turbine blades, connector elements and space vehicles referring to the recent requirements in developing technology. These needs are required to a wide range of curved element (elliptic, parabola, catenary, cycloid, and circle) having curvature range along the arc length with different section shapes and sizes. Curved beams can either be solved analytically or numerically. Some of the well-known numerical procedures are series solution, carryover matrix method and finite element analysis. Due to the limitations of analytical solution procedure in application finite element procedure is applied in this study. In finite elements analysis either displacement or mixed type elements are used. In mixed type elements, besides the displacements and rotations, also forces and moments are the nodal unknowns. Depending on the necessity of the problem mainly two different groups of field equations are used for deriving functional, necessary for the finite element formulations. Timoshenko beam theory considers the effect of shear deformation and rotary inertia while Bernoulli-Navier neglects them. This necessity mainly depends on the dimensions of the cross-section of the beam member. Static analysis are presented in the following non-circular curves having variable cross-section: semi-circular and semi-elliptical arches, elliptic-helical beam in Gimena et al. (2008); circular, semi-elliptical, parabolic in Tufekci et al. (2017). Huang et al. (1998c) are developed an exact solution for in-plane vibration arches having variable curvature and cross-section. A convergence analysis for the natural frequencies of the semi-elliptical beam is obtained and compared with the literature. Then, the influence of some geometric parameters on the natural frequencies is presented for parabolic arches.

[1]Research Assistant, Faculty of Civil Engineering, Istanbul Technical University, Istanbul, Turkey, ermism@itu.edu.tr
[2]Research Assistant, Faculty of Civil Engineering, Istanbul Technical University, Istanbul, Turkey, umiaribas@itu.edu.tr
[3]Professor, Faculty of Civil Engineering, Istanbul Technical University, Istanbul, Turkey, eratli@itu.edu.tr
[4]Professor, Faculty of Civil Engineering, Istanbul Technical University, Istanbul, Turkey, omurtagm@itu.edu.tr

Huang et al. (2000) are studied the linear out-of-plane dynamic responses of non-circular plane curves having variable cross-section by extending previous works about uniform curved beams (Huang et al. 1998a) and in-plane dynamic responses (Huang et al. 1998b). Free vibration analysis are investigated in the following non-circular plane curves: parabolic, elliptic, sinusoidal, catenary types in Oh et al. (1999) and Oh et al. (2000); parabolic, elliptical, sinusoidal in Yang et al. (2008); elliptical arch in Shahba et al. (2013); parabolic, elliptic, Tschirnhausen′s cubic in Luu et al. (2015); horseshoe elliptic in Lee et al. (2016). The free vibration analysis of composite laminated and sandwich circular and non-circular beams is studied in Ye et al. (2016). Free vibration and stability analysis of elliptic beam is considered in Nieh et.al. (2003). Rajasekaran (2013) solved the static, stability, free and forced vibrations of axially functionally graded tapered circular and non-circular arches by using finite element method.
In this study, the free vibration analysis of the planar elliptical Timoshenko beam having three different cross-sections (two different elliptically oriented cross sections and circular cross-section) is performed via mixed finite element method. The exact formulation of arc length and curvature of an elliptical plane curve is derived by using the formulation given in Ermis and Omurtag (2017) and the influence of some parameters (e.g. cross-sections, the ratio of the minimum radius of elliptical beam to the maximum radius of elliptical beam, the opening angle) on the natural frequencies of the planar elliptical beam are investigated. As far as the knowledge of the authors, studies on elliptical plane beams are rare and this study is a contribution to the literature about research on elliptical beams by mixed finite element method.

2. FORMULATION

2.1 Field Equations and Functional

The field equations for the spatial beams, which are based on the Timoshenko beam theory and refer to the Frenet coordinate system, are discussed in Omurtag and Aköz (1992) and applied to the free vibration problem of the helicoidal bars having non-circular cross-sections in Eratlı et al. (2016). Using $\mathbf{u} = u_t\,\mathbf{t} + u_n\,\mathbf{n} + u_b\mathbf{b}$ is the displacement vector, $\mathbf{\Omega} = \Omega_t\,\mathbf{t} + \Omega_n\,\mathbf{n} + \Omega_b\mathbf{b}$ is the rotational vector, $\mathbf{T} = T_t\,\mathbf{t} + T_n\,\mathbf{n} + T_b\mathbf{b}$ is the force vector, $\mathbf{M} = M_t\,\mathbf{t} + M_n\,\mathbf{n} + M_b\mathbf{b}$ is the moment vector, ρ is the density of material, A is the area of the cross-section, $\mathbf{I} = I_t\,\mathbf{t} + I_n\,\mathbf{n} + I_b\mathbf{b}$ is the moment of inertia vector, $\mathbf{C}$ is the compliance matrix, $\mathbf{q}$ and $\mathbf{m}$ are the distributed external force vector and moment vector, respectively. The field equations can be written in the form.

$$\begin{aligned} -\mathbf{T}_{,s} - \mathbf{q} + \rho A\ddot{\mathbf{u}} &= \mathbf{0} \\ -\mathbf{M}_{,s} - \mathbf{t}\times\mathbf{T} - \mathbf{m} + \rho\mathbf{I}\,\ddot{\mathbf{\Omega}} &= \mathbf{0} \\ \mathbf{u}_{,s} + \mathbf{t}\times\mathbf{\Omega} - \mathbf{C}_\gamma\mathbf{T} &= \mathbf{0} \\ \mathbf{\Omega}_{,s} - \mathbf{C}_\kappa\mathbf{M} &= \mathbf{0} \end{aligned} \tag{1}$$

where the accelerations are denoted by $\ddot{\mathbf{u}} = \partial^2\mathbf{u}/\partial t^2$, $\ddot{\mathbf{\Omega}} = \partial^2\mathbf{\Omega}/\partial t^2$. Equation (1) can be written in operator form as $\mathbf{Q} = \mathbf{L}\mathbf{y} - \mathbf{f}$, if the operator is potential, the equality $\langle \mathrm{d}\mathbf{Q}(\mathbf{y},\bar{\mathbf{y}}),\mathbf{y}^*\rangle = \langle \mathrm{d}\mathbf{Q}(\mathbf{y},\mathbf{y}^*),\bar{\mathbf{y}}\rangle$ must be satisfied (Oden and Reddy 1976). $\mathrm{d}\mathbf{Q}(\mathbf{y},\bar{\mathbf{y}})$ and $\mathrm{d}\mathbf{Q}(\mathbf{y},\mathbf{y}^*)$ are Gâteaux derivatives of the operator in the directions of $\bar{\mathbf{y}}$ and $\mathbf{y}^*$, respectively. After proving the operator to be potential and considering the harmonic motion of the helix in the free vibration analysis (and also $\mathbf{q} = \mathbf{m} = \mathbf{0}$), the functional yields to the following form

$$\begin{aligned} \mathbf{I}(\mathbf{y}) = &-[\mathbf{u},\mathbf{T}_{,s}] + [\mathbf{t}\times\mathbf{\Omega},\mathbf{T}] - [\mathbf{M}_{,s},\mathbf{\Omega}] - \tfrac{1}{2}[\mathbf{C}_\kappa\mathbf{M},\mathbf{M}] - \tfrac{1}{2}[\mathbf{C}_\gamma\mathbf{T},\mathbf{T}] - \tfrac{1}{2}\rho A\omega^2[\mathbf{u},\mathbf{u}] - \tfrac{1}{2}\rho\omega^2[\mathbf{I}\mathbf{\Omega},\mathbf{\Omega}] \\ &+ \left[\left(\mathbf{T}-\hat{\mathbf{T}}\right),\mathbf{u}\right]_\sigma + \left[\left(\mathbf{M}-\hat{\mathbf{M}}\right),\mathbf{\Omega}\right]_\sigma + [\hat{\mathbf{u}},\mathbf{T}]_\varepsilon + \left[\hat{\mathbf{\Omega}},\mathbf{M}\right]_\varepsilon \end{aligned} \tag{2}$$

where ω is the natural angular frequency and the square parentheses indicate the inner product. The terms with hats in Equation (2) are known values on the boundary and the subscripts ε and σ represent the geometric and the dynamic boundary conditions, respectively.

2.2 Mixed Finite Element Method and the Free Vibration Analysis

A two-nodded curved element is employed to discretize the beam domain. The curved element has 2×12 degrees of freedom. The variable vectors per node are $\mathbf{u}, \mathbf{\Omega}, \mathbf{T}, \mathbf{M}$. Linear shape functions are employed for the interpolation. The curvatures are satisfied exactly at the nodal points and linearly interpolated through the element (Ermis and Omurtag 2017). The problem of determining the natural frequencies of a structural system reduces to the solution of a standard eigenvalue problem $([\mathbf{K}]-\omega^2[\mathbf{M}])\{\mathbf{u}\}=\{\mathbf{0}\}$ where $[\mathbf{K}]$ is the system matrix, $[\mathbf{M}]$ is the mass matrix for the entire domain, $\mathbf{u}$ is the eigenvector and ω is the natural angular frequency of the system. Hence the explicit form of standard eigenvalue problem in the mixed formulation is

$$\left(\begin{bmatrix}[\mathbf{K}_{11}] & [\mathbf{K}_{12}]\\ [\mathbf{K}_{22}] & [\mathbf{K}_{22}]\end{bmatrix}-\omega^2\begin{bmatrix}[\mathbf{0}] & [\mathbf{0}]\\ [\mathbf{0}] & [\mathbf{M}]\end{bmatrix}\right)\begin{Bmatrix}\{\mathbf{F}\}\\ \{\mathbf{U}\}\end{Bmatrix}=\begin{Bmatrix}\{\mathbf{0}\}\\ \{\mathbf{0}\}\end{Bmatrix} \tag{3}$$

where $\{\mathbf{F}\}$ denotes the nodal force and the moment vectors and $\{\mathbf{U}\}=\{\mathbf{u}\ \mathbf{\Omega}\}^T$ signifies the nodal displacement and rotation vectors. The $\{\mathbf{F}\}$ vector is eliminated in Equation (3) and the eigenvalue problem in the mixed formulation becomes $([\mathbf{K}^*]-\omega^2[\mathbf{M}])\{\mathbf{U}\}=\{\mathbf{0}\}$ where the condensed system matrix is $[\mathbf{K}^*]=[\mathbf{K}_{22}]-[\mathbf{K}_{12}]^{\mathrm{T}}[\mathbf{K}_{11}]^{-1}[\mathbf{K}_{12}]$.

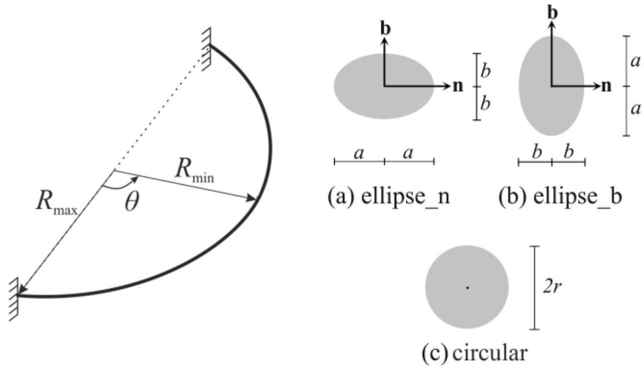


Figure 1. A planar elliptical beam and the types of cross-sections

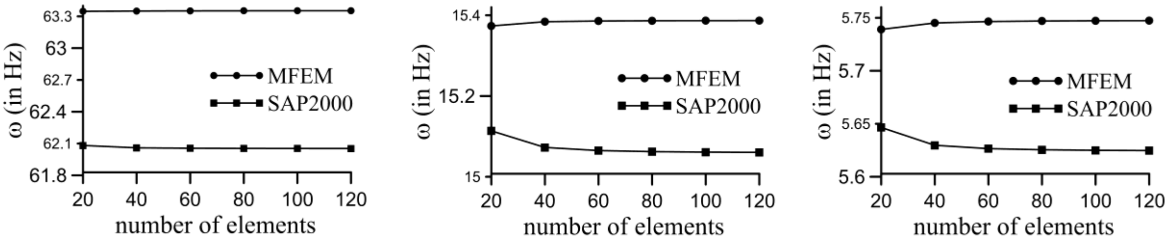
3. NUMERICAL EXAMPLES

This is a parametric analysis of a planar elliptical beam. Through the analysis, different cross-sections (two different elliptical orientations and circular), the minimum radius of elliptical beam to the maximum radius of elliptical beam ratios (R_{min}/R_{max}), the opening angles (θ) are handled. The influence of these parameters on the natural frequency of the planar elliptical beam is investigated. The planar elliptical beam is clamped at both ends. The orientations of the two different elliptical cross sections are as shown in Figure 1. The abbreviations "ellipse_n" and "ellipse_b" are used elliptical cross sections with long side oriented horizontal and vertical direction, respectively. The common parameters for the examples are: the modulus of elasticity of the beam is $E=210\,\mathrm{GPa}$, Poisson's ratio is $\upsilon=0.3$, the density of material is $\rho=7850\,\mathrm{kg/m^3}$, the maximum radius of elliptical beam is $R_{max}=2\,\mathrm{m}$ and R_{min}/R_{max} ratios are $0.25, 0.5, 0.75, 0.9999$. The opening angles are $\theta=90°, 180°, 270°$. The net areas of the all three cross-sectional geometries (ellipse_n, ellipse_b, circular) are equal to each other and the dimensions of the elliptical and circular cross-sections are $a=6\,\mathrm{cm}$, $b=3\,\mathrm{cm}$, $r=4.24264\,\mathrm{cm}$ (see

Figure 1).

3.1 Convergence Test and Comparison

The free vibration analysis of the planar elliptical beam having circular cross-section for $\theta = 90°, 180°, 270°$ is carried out using 20, 40, 60, 80, 100 and 120 mixed finite elements. The minimum radius of elliptical beam to the maximum radius of elliptical beam ratio is $R_{min} / R_{max} = 0.5$. The mixed finite element results are compared with the commercial program SAP2000 for the fundamental frequencies (ω) and the results are presented in Figure 2. The normalized percent difference respect to the mixed finite element results between two finite element models is $\sim 2\%$. In the following examples, 100 elements are employed.



(a) $\theta = 90°$ (b) $\theta = 180°$ (c) $\theta = 270°$

Figure 2. Convergence test and comparisons for three different opening angles

3.2 Parametric Study

The natural frequencies of the planar elliptical beam having three different (ellipse_n, ellipse_b, circular) cross-sections are obtained via the mixed finite element for different R_{min} / R_{max} ratios $(0.25, 0.5, 0.75, 0.9999)$ and θ opening angles $(90°, 180°, 270°)$. The first three natural frequencies of the planar elliptical beam are given in Tables 1-4 for $R_{min} / R_{max} = 0.25, 0.5, 0.75, 0.9999$, respectively.

Table 1. The natural frequencies of beams ($R_{min} / R_{max} = 0.25$). Different cross-sections and opening angles.

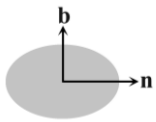
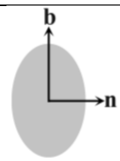
θ	**mode**	**ω (in Hz)**		
		 **ellipse_n**	 **circular**	 **ellipse_b**
90°	1	59.865	83.108	112.057
	2	162.992	226.833	309.131
	3	316.462	438.917	596.335
180°	1	14.981	20.649	26.910
	2	41.307	57.246	76.655
	3	80.219	111.822	151.899
270°	1	6.224	8.335	10.311
	2	14.021	19.066	24.826
	3	34.103	44.814	56.018

Table 2. The natural frequencies of beams ($R_{min} / R_{max} = 0.5$). Different cross-sections and opening angles.

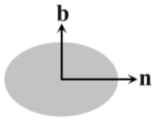
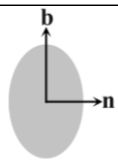
θ	mode	ω (in Hz)		
				
		ellipse_n	**circular**	**ellipse_b**
	1	45.446	63.352	86.248
90°	2	124.589	174.070	238.780
	3	245.093	341.313	466.689
	1	11.187	15.387	19.887
180°	2	31.038	43.073	57.779
	3	60.457	84.611	115.738
	1	4.342	5.747	6.983
270°	2	10.512	14.402	18.802
	3	23.900	32.469	41.945

Table 3. The natural frequencies of beams ($R_{min} / R_{max} = 0.75$). Different cross-sections and opening angles.

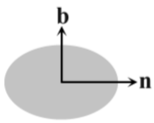
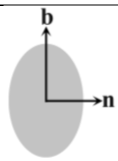
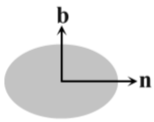
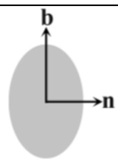
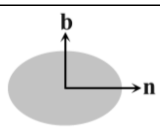
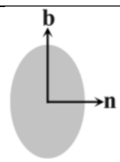
θ	mode	ω (in Hz)		
				
		ellipse_n	**circular**	**ellipse_b**
	1	33.843	47.339	64.844
90°	2	95.183	133.092	182.367
	3	188.981	263.810	361.550
	1	7.947	10.998	14.418
180°	2	22.410	31.157	41.748
	3	45.257	63.231	85.905
	1	3.148	4.198	5.104
270°	2	8.051	11.087	14.438
	3	17.856	24.560	31.996

Table 4. The natural frequencies of beams ($R_{\min} / R_{\max} = 0.9999$). Different cross-sections and opening angles.

θ	mode	ω (in Hz)		
		 **ellipse_n**	 **circular**	 **ellipse_b**
90°	1	25.942	36.323	49.840
	2	73.716	103.164	141.415
	3	146.614	205.079	281.973
180°	1	5.705	7.938	10.575
	2	16.489	22.878	30.450
	3	34.426	47.952	64.470
270°	1	2.401	3.211	3.905
	2	6.235	8.595	11.181
	3	13.723	18.931	24.720

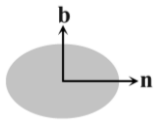
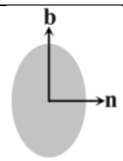
The natural frequencies decrease by increasing $R_{\min} / R_{\max}$ ratios for each value of $\theta = 90°, 180°, 270°$ and all type of cross-sections (see Tables 1-4). The fundamental natural frequency values of $R_{\min} / R_{\max} = 0.5, 0.75, 0.9999$ are normalized with respect to $R_{\min} / R_{\max} = 0.25$ that correspond $\theta = 90°, 180°, 270°$ and the percent reductions are given in Table 5 for all type of cross-sections.

Table 5. The percent reductions for the fundamental natural frequencies of beam having circular and elliptical cross sections in the case of $R_{\min} / R_{\max} = 0.5, 0.75, 0.9999$ with respect to $R_{\min} / R_{\max} = 0.25$

θ	$R_{\min} / R_{\max}$	 **ellipse_n**	 **circular**	 **ellipse_b**
		diff. % = $[(R_{\min}/R_{\max})_{0.25} - (R_{\min}/R_{\max})_i) \times 100 / (R_{\min}/R_{\max})_{0.25}]$ i=0.5, 0.75, 0.9999		
90°	0.5	24.1	23.8	23.0
	0.75	43.5	43.0	42.1
	0.9999	56.7	56.3	55.5
180°	0.5	25.3	25.5	26.1
	0.75	47.0	46.7	46.4
	0.9999	61.9	61.6	60.7
270°	0.5	30.2	31.1	32.3
	0.75	49.4	49.6	50.5
	0.9999	61.4	61.5	62.1

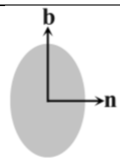
As the opening angle increases, the decreasing trend is observed for the natural frequencies of the planar elliptical beam. The percent reductions of the fundamental natural frequencies of the planar elliptical beam having opening angle $(\theta = 90°)$ for circular and elliptical cross-sections and $R_{min} / R_{max} = 0.25, 0.5, 0.75, 0.9999$ ratios according to opening angles $(\theta = 180°, 270°)$ are given in Table 6.

Table 6. The percent reductions for the fundamental natural frequencies of beam having circular and elliptical cross sections in the case of $\theta = 180°, 270°$ with respect to $\theta = 90°$.

R_{min} / R_{max}	θ	 **ellipse_n**	 **circular**	 **ellipse_b**
		diff. % = [($\theta_{90°}$ - θ_i)×100/ $\theta_{90°}$], i=180°, 270°		
0.25	180°	75.0	75.2	76.0
	270°	89.6	90.0	90.8
0.5	180°	75.4	75.7	76.9
	270°	90.4	90.9	92.0
0.75	180°	76.5	76.7	77.8
	270°	90.7	91.1	92.1
0.9999	180°	78.0	78.1	78.8
	270°	90.7	91.2	92.2

If the natural frequencies of the planar elliptical beam having ellipse_n cross-section are compared with the natural frequencies of beam having circular and ellipse_b cross-sections, an increasing trend is observed. The percent increases in the fundamental natural frequencies are tabulated in Table 7.

Table 7. The percent increases for the fundamental natural frequencies of beam having circular and ellipse_b cross sections with respect to ellipse_n

R_{min} / R_{max}	θ	 **circular**	 **ellipse_b**
		diff. % = [ellipse_n - (circular or ellipse_b)×100/ellipse_n]	
0.25	90°	-38.8	-87.2
	180°	-37.8	-79.6
	270°	-33.9	-65.7
0.5	90°	-39.4	-89.8
	180°	-37.5	-77.8
	270°	-32.4	-60.8
0.75	90°	-39.9	-91.6
	180°	-38.4	-81.4
	270°	-33.4	-62.1
0.9999	90°	-40.0	-92.1
	180°	-39.1	-85.4
	270°	-33.7	-62.6

4. CONCLUSIONS

A parametric study for the planar elliptical Timoshenko beam is carried out via mixed finite element method to investigate the influence of some geometric parameters on the natural frequencies of the beam. Then, it is solved using exact curvatures at the nodal points and their interpolations through the element. As a convergence test, a semi-elliptical beam having circular cross-section for $R_{\min} / R_{\max} = 0.5$ is handled, results of the mixed finite element program is compared by the commercial program SAP2000 and an excellent agreement is achieved. Some examples are solved to investigate the influence of the some parameters (cross-sections: ellipse_n, ellipse_b, circular, the ratio of the minimum radius of elliptical beam to the maximum radius of elliptical beam: $R_{\min} / R_{\max}$, the opening angle: θ) on the free vibration analysis of the planar elliptical beam. Following remarks can be cited:

- As $(R_{\min} / R_{\max})$ ratio increases, a reduction is observed for the natural frequencies (see Tables 1-5).
- An increase for the opening angle of the planar elliptical beam caused a reduction of the natural frequencies (Tables 1-4, Table 6).
- If the natural frequencies of the elliptical beam having ellipse_n cross-section are compared with the natural frequencies of beam having circular and ellipse_b cross-sections, an increasing in the natural frequencies is observed (Tables 1-4, 7).

5. ACKNOWLEDGMENTS

This research is supported by the Research Foundation of ITU under project no MGA-2017-40739. This support is gratefully acknowledged by the authors.

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