HYBRID ASYNCHRONOUS ABSORBING LAYERS FOR SEISMIC WAVE PROPAGATION IN 2D UNBOUNDED DOMAINS

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ABSTRACT

One of the critical points in numerical simulation for wave propagation problems in unbounded domains is how to simulate infinite media. Two types of absorbing layers are proposed in this paper for reproducing unbounded medium, which are Kosloff absorbing layer and Rayleigh absorbing layer. Firstly, by using strong forms of elastic wave propagation, the absorbing abilities of absorbing layers related to target performance criterion are derived in form of logarithmic decrement. The general formulas for designing Kosloff and Rayleigh absorbing layer are proposed. In addition, no-reflectio condition are also obtained to avoid the spurious waves reflected at the interface. In the case of non-harmonic waves, Kosloff absorbing layer is proved to be independent of frequency in terms of attenuation and less sensitive to design parameters in comparison to Rayleigh absorbing layer. Secondly, the introduction of damping will significantly reduce the stable time increment in full-explicit scheme. Hence, a subdomain strategy is adopted, enabling to consider implicit scheme in absorbing layer subdomain with a large time step, independently of the explicit scheme employed in the physical subdomain. Two subdomains have been coupled to carry out explicit/implicit co-computations based on dual Schur approach. The proposed absorbing region is called Hybrid (different time integrators) Asynchronous (different time steps) Absorbing Layers. Then, Lamb’s test is implemented to illustrate efficiency of two approaches in terms of accuracy and CPU time, in comparison to hybrid asynchronous Perfectly Matched Layers (PML). The difference of three approaches is analyzed to demonstrate the advantages and disadvantages of each approach.

Keywords: Hybrid asynchronous, absorbing layers, co-simulation, wave propagation, subdomain decomposition

1. INTRODUCTION

One of the critical points of the numerical simulation of wave propagation problems in unbounded domains using the finite element method is how to simulate infinite media. The simplest way is to consider a very large extended numerical mesh, but it leads to important computation times, in particular when long time duration excitations are considered. Hence, non-reflective boundary conditions are required at the boundary of the truncated domain for mimicking infinite or semi-infinite media. Several kinds of artificial boundaries in numerical methods have been developed to avoid spurious waves reflected at the boundary, such as the infinite elements (Bettess 1977, Su and Wang 2013), absorbing boundary conditions (Enquist et al. 1977), or PML (Perfect Matched Layers).

PML proposed by Bérenger (1994) is becoming increasingly used for dealing with infinite media in the context of finite difference and finite element methods. The spatial (FEM) and time discrete formulation of PML proposed by Basu and Chopra (2004) has been recently reemployed by Brun et al. (2016) for extending the PML capabilities towards hybrid time integration and multi time steps. However, PML is not simple to be implemented. Semblat et al. (2011) and Rajagopal et al. (2012) introduced more convenient techniques for

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implementing efficient absorbing conditions into commercial finite element codes, called Absorbing Layers using Increasing Damping (ALID), based on Rayleigh viscous damping matrix associated with an increasing damping ratio in the thickness of the absorbing region. More recently, Zafati et al. (2015, 2014) proposed a procedure for designing ALID with some improvements. The disadvantage of Rayleigh absorbing layer is that the absorbing ability is not independent of frequency. Consequently, it is interesting to exploit a kind of absorbing layer which can be implemented easily like Rayleigh layer and has the independence of frequency like PML. Kosloff absorbing boundary is proposed by D. Kosloff, and R. Kosloff (1986), based on a simple modification in the wave propagating equation. In Kosloff medium, the wave travels without changing shape and the wave amplitude decreases with distance at a frequency independent rate. Some similarities between PML and Kosloff absorbing boundary have been discussed by J. M. Carcione and D. Kosloff (2013). Therefore, Kosloff absorbing layer is investigated in this paper in order to assess its relevance as an efficient and convenient absorbing layer for seismic wave propagation.

Firstly, the design of Kosloff absorbing layer is proposed by using the strong form of elastic wave propagation in Kosloff medium. The absorbing ability of Kosloff absorbing layer associated with a performance criterion are derived in the form of logarithmic decrement and proved to be independent of frequency. A general formula for designing Kosloff absorbing layer using a multi-layer strategy is proposed. In addition, no-reflection conditions are obtained to avoid the spurious waves reflected at the interface between physical domain and Kosloff absorbing layer domain.

Kosloff absorbing layer is proved to be independent of frequency in terms of attenuation and is found to be less sensitive to the layer design parameters in comparison to Rayleigh absorbing layer. Secondly, the weak formulation of the decomposed problem is obtained in order to derive the discretization in 2D and 3D space and for the time discretization. The GC method proposed by Gravouil and Combescure is employed by adopting the continuity of velocities at the interface to couple an independent explicit integrator for the subdomain of interest with a different implicit one for the absorbing layers based on the dual Schur approach. The derived absorbing region is called Hybrid (different time integrators) Asynchronous (different time steps) Absorbing Layers. Finally, Lamb’s test have been implemented to illustrate the efficiency of two hybrid asynchronous approaches in terms of accuracy and CPU time, in comparison to the hybrid asynchronous PML (Brun et al. 2016).

2. STRONG FORM FOR THE WAVE PROPAGATION IN A KOSLOFF MEDIUM

The design of Kosloff absorbing layer aims at damping out all the incident waves from the domain of interest while minimizing the spurious waves reflected at the boundary of the truncated domain. For this purpose, with the strong form for the wave propagation in the Kosloff medium, the absorbing ability of Kosloff absorbing layer related to a target performance criterion will be quantified in the form of logarithmic decrement. The optimal conditions at the interface between a non-dissipative elastic medium $\Omega_1$ and a dissipative Kosloff medium $\Omega_2$ will be analytically established by considering the continuous problem of wave propagation. The displacement vector field $\mathbf{u}$ in the Kosloff domain $\Omega_2$ is governed by Equations 1, 2 and 3:

$$\rho \partial_t^2 \mathbf{u} = \text{div}(\sigma(\mathbf{u})) - 2\rho \gamma \partial_t \mathbf{u} - \rho \gamma^2 \mathbf{u}$$  \hspace{1cm} (1)

$$\sigma = \lambda \text{tr}(\varepsilon(\mathbf{u})) + 2\mu \varepsilon(\mathbf{u})$$  \hspace{1cm} (2)

$$\varepsilon = \frac{1}{2} \left[ \text{grad}(\mathbf{u}) + \text{grad}(\mathbf{u})^T \right]$$  \hspace{1cm} (3)
Equations 1 to 3 constitute the strong form of the propagation in a Kosloff medium, $\sigma, \varepsilon, \lambda, \mu, \rho, \gamma$ being the stress matrix, strain matrix, Lamé’s coefficients, the density and damping ratio related to the Kosloff domain $\Omega_2$, respectively.

2.1 The absorbing ability of Kosloff medium

The argument is developed for 1D wave propagation problem by distinguishing the P-waves and the S-waves in their strong form, written as (Equations 4 and 5):

$$\rho \partial_t^2 u = (\lambda + 2\mu) \partial_x^2 u - 2\rho \gamma \partial_t u - \rho \gamma^2 u \quad P - waves$$

$$\rho \partial_t^2 u = \mu \partial_x^2 u - 2\rho \gamma \partial_t u - \rho \gamma^2 u \quad S - waves$$

By introducing harmonic solutions $u(x, t) = u_0 \exp(i(\omega_0 t - kx))$ into wave propagation equations, the expression of the wave number $k$ can be obtained (Equation 6):

$$k = \frac{\omega_0}{v} \left(1 - i \frac{\gamma}{\omega_0}\right)$$

with the velocity $v$ equal to $\sqrt{\frac{\lambda + 2\mu}{\rho}}$ for P-wave and $\sqrt{\frac{\mu}{\rho}}$ for S-wave. The expression of the propagating wave in the 1D Kosloff medium is shown below (Equation 7):

$$u(x, t) = u_0 \exp(i(\omega_0 t - kx)) \exp\left(-\frac{\gamma x}{v}\right)$$

Using the previous expression of the propagating wave, the relationship between the logarithmic decrement, the thickness and the damping ratio of the Kosloff absorbing layer is derived by (Equations 8):

$$\delta = \ln\left(\frac{|u(x)|}{|u(x + \Delta x)|}\right) = \frac{\gamma \Delta x}{v_p}$$

It can be seen that the frequency wave $\omega_0$ has no influence on the absorbing ability of Kosloff absorbing layer with regard to the logarithmic decrement, which means that all frequencies can be attenuated in the same way. Thus, Kosloff damping turns out to be independent of frequency. The velocity of P waves is bigger than the velocity of S waves in the same medium. In other words, based on the above relationships, in order to reach the same logarithmic decrement, the necessary layer thickness for damping out S waves is smaller than the one related to P waves. For the design of absorbing layer, $v_p$ will be chosen to make sure that all the waves can be attenuated according to the target decrement.

2.2 No-reflection condition at the interface

Wave propagation problem from an elastic medium to a Kosloff medium is considered below in the case of 1D harmonic wave by three components (Equations 9, 10 and 11), the incident wave, the transmitted wave, and the reflected wave, as shown in Figure 1.

![Figure 1. Wave propagation from elastic medium to Kosloff medium.](image-url)
Based on the continuity of displacements and equilibrium of stresses at the interface (Equations 12 and 13),

\[ u_2(x, t) = T \exp \left[i \omega_0 \left(-\frac{x}{v_2}\right)\right] \exp \left[-\frac{y}{v_2}\right] \]

\[ u_R(x, t) = R \exp \left[i \omega_0 (t + \frac{x}{v_1})\right] \]  

(9)  

(10)  

(11)

with \( k_2 \) equal to \( \lambda_1 + 2\mu_1 \) for P-waves or \( 2\mu_1 \) for S-waves while \( k_2 \) equal to \( \lambda_2 + 2\mu_2 \) for P-waves or \( 2\mu_2 \) for S-waves. The reflection coefficient \( R_{\text{interface}} \) at the interface can be obtained as (Equations 14 and 15),

\[ \frac{R}{A} = \frac{-\rho_2 v_2 \left(1 - i \frac{Y}{\omega_0}\right) + \rho_1 v_1}{\rho_2 v_2 \left(1 - i \frac{Y}{\omega_0}\right) + \rho_1 v_1} \]  

(14)

By setting \( \alpha = \frac{\rho_2 v_2}{\rho_1 v_1} \),

\[ \frac{R}{A} = \frac{1 - \alpha \left(1 + i \frac{Y}{\omega_0}\right)}{1 + \alpha \left(1 - i \frac{Y}{\omega_0}\right)} \]  

(15)

If we want to have no reflection at the interface, reflection coefficient should be zero, the relationship without reflections at the interface is given below by setting the real part of reflection coefficient equal to zero (Equation 16),

\[ \alpha = \frac{1}{\sqrt{1 + \left(\frac{Y}{\omega_0}\right)^2}} \]  

(16)

Now, we can give a simple condition on material properties of the Kosloff medium so as to cancel the reflected waves in the case of harmonic waves (Equation 17),

\[
\begin{aligned}
E_2 &= \frac{E_1}{1 + \left(\frac{Y}{\omega_0}\right)^2} \\
\rho_2 &= \rho_1
\end{aligned}
\]  

(17)

where \( E_1 \) and \( E_2 \) are Young’s moduli, \( \rho_1 \) and \( \rho_2 \) are the densities of subdomains \( \Omega_1 \) and \( \Omega_2 \), respectively. The relationship shows that though the Kosloff medium is independent of frequency in terms of decrement, the non-reflection condition at the interface depends on the frequency. Therefore, for non-harmonic waves, there will be still some reflections at the interface, the influence of chosen \( f_0 \) will be evaluated in the following.
2.3 Design of Kosloff absorbing layer using a multi-layer strategy

The Absorbing Layers using Increasing Damping, called ALID, proposed by Semblat et al. (2011) and Rajagopal et al. (2012), as shown in Figure 2, is considered by tuning the elastic parameters of each layer depending on the selected absorbing coefficient as given by the optimal conditions in Equation (17). The main idea is to divide the Kosloff absorbing medium into several uniform layers, so that the decrements produced by each layer can be multiplied. Because of the logarithmic form of decrement, the total logarithmic decrement can be easily obtained. The evolution of damping ratio in layers has an important influence on the efficiency of the ALID. In this paper, a nonlinear increase of damping ratio is adopted to achieve a better accuracy.

![Figure 2](image.png)

Figure 2. Evolution of the damping ratios in multi-layer absorbing subdomain

The parameters of each layer satisfying the optimal conditions at each interface are given by (Equation 18):

\[
\begin{aligned}
E_2^{(i+1)} &= \frac{1 + \left(\frac{Y_i}{\omega_0}\right)^2}{1 + \left(\frac{Y_{i+1}}{\omega_0}\right)^2} E_2^{(i)} \\
E_2^{(i)} &= \frac{1}{1 + \left(\frac{Y_i}{\omega_0}\right)^2} E_1 \\
v_2^{(i)} &= v_1 \\
\rho_2^{(i)} &= \rho_1 \\
Y_i &= \gamma_0 \left(\frac{x}{L}\right)^m
\end{aligned}
\]

(18)

where \(E_2^{(i)}\) the Young’s modulus, \(Y_i\) the damping ratio, \(v_2^{(i)}\) Poisson’s ratio, \(\rho_2^{(i)}\) the density of each layer \(i\) in the subdomain \(\Omega_2\), \(\omega_0\) the chosen frequency to design absorbing layers, \(\gamma_0\) the damping ratio in the last layer, \(m\) the index of the evolution function, \(x\) distance in the thickness of absorbing layers, \(L\) the thickness of all the absorbing layers. The total logarithmic decrement \(\delta\) is written as follows (Equation 19):

\[
\begin{aligned}
\delta_i &= \gamma_0 \left(\frac{x}{L}\right)^m \frac{\Delta x}{v_p} \\
\delta &= \int_0^L \gamma_0 \left(\frac{x}{L}\right)^m \frac{dx}{v_p} = \frac{\gamma_0 L}{(m + 1)v_p}
\end{aligned}
\]

(19)

where \(\delta_i\) represents the logarithmic decrement of each sublayer \(i\), \(\Delta x\) represents the thickness of each layer. The attenuation coefficient \(R_{\text{attenuation}}\) for the system of absorbing layers is defined by (Equation 20):
For example, if the goal is to reach a target logarithmic decrement $\delta=\ln(10)$, this means that 90% of the amplitude of the incident will be absorbed from the interface to the end of the damping layers. Next, the attenuation also occurs for the reflection process from the end of the damping layer towards the interface. Thus the incident wave is attenuated by 99%. Hence, the attenuation coefficient $R_{\text{attenuation}}$ is theoretically equal to 1%. Finally, we can propose the general formula to design Kosloff absorbing layers as shown below. When the $R_{\text{attenuation}}$, the total thickness $L$, the evolution of damping ratio are chosen, $C_p$ is the velocity of P waves, $y_0$ can be obtained by the formula. After that, all the parameters for designing the absorbing layers can be determined (Equation 21).

$$y_0 = \frac{(m + 1)}{2L} \times C_p \times \ln \left( \frac{1}{R_{\text{attenuation}}} \right)$$  \hspace{1cm} (21)

It is important to remark that the general formula to design Kosloff absorbing is similar to that of PML widely used proposed by Collino and Tsogka (2001), based on one-dimensional wave propagation ideas, $\beta_0$ is the damping parameter in the last PML layer (Equation 22).

$$\beta_0 = \frac{(m + 1)}{2L} \times C_p \times \ln \left( \frac{1}{R_{\text{attenuation}}} \right)$$  \hspace{1cm} (22)

3. THE SPACE AND TIME DISCRETIZATION FOR HA-KOSLOFF AND HA-RAYLEIGH

The elastic wave propagation from an elastic non-dissipative medium to a dissipative medium should be discretized in space and in time. The case of HA-Kosloff is introduced. The space and time discretization for HA-Rayleigh can be obtained by the similar method. Let $\Omega$ be a bounded domain belonging to $\mathbb{R}^2$ with a regular boundary. $J=\lbrack 0, T \rbrack$ is the time interval of interest. The domain $\Omega$ is divided into two partitions $\Omega_1$ and $\Omega_2$, as shown in Figure 3, such as: $\Omega_1 \cap \Omega_2 = \emptyset$ and $\partial \Omega_1 \cap \partial \Omega_2 = \Gamma_I$. $\Gamma_I$ denotes the interface between the two subdomains, subdomain $\Omega_1$ representing the non-dissipative medium (the domain of interest) and subdomain $\Omega_2$ the Rayleigh medium.

Figure 3. Domain $\Omega$ divided into two subdomains $\Omega_1$ and $\Omega_2$ with their material characteristics. Subdomain $\Omega_1$ is handled by an explicit scheme with fine time step and subdomain $\Omega_2$ by an implicit scheme with large time step

3.1 Weak form and space discretization

The subdomain $\Omega_1$ is characterized by its density $\rho_1$, Young’s modulus $E_1$, Poisson coefficient $\nu_1$, $b_1$ the body force, $u_1^D$ the Dirichlet prescribed displacement on $\Gamma_1^D$ and $g_1^N$ the traction force at the Neumann condition on $\Gamma_1^N$. The subdomain $\Omega_2$ is characterized by its density $\rho_2$, Young’s modulus $E_2$, Poisson coefficient $\nu_2$, $b_2$ the...
body force \( u_x^D \), the Dirichlet prescribed displacement on \( \Gamma_2^D \), \( g_2^N \) the traction force at the Neumann condition on \( \Gamma_2^N \) and the parameter \( y \) introduced in the strong form of the wave equation.

In order to write the weak form of the coupled problem in \( \Omega \) divided into two partitions \( \Omega_1 \) and \( \Omega_2 \), test functions \( \psi_1 \) and \( \psi_2 \) belonging to the appropriate spaces \( W_1^* \) and \( W_2^* \) must be introduced (Equation 23):

\[
\begin{align*}
\left\{ \psi_1 \left( X_1 \right) \right\} & \in W_1^* , W_1^* = \left\{ \psi_1 \in \left( H^1 \left( \Omega_1 \right) \right)^d \text{ and } \psi_1 = 0 \text{ on } \Gamma_1^D \right\} \\
\left\{ \psi_2 \left( X_2 \right) \right\} & \in W_2^* , W_2^* = \left\{ \psi_2 \in \left( H^1 \left( \Omega_2 \right) \right)^d \text{ and } \psi_2 = 0 \text{ on } \Gamma_2^D \right\}
\end{align*}
\]  

(Equation 23)

The solutions \( u_1 \) and \( u_2 \) belong to the appropriate spaces \( W_1 \) and \( W_2 \) (Equation 24):

\[
\begin{align*}
\left\{ u_1 \left( X_1 , t \right) \right\} & \in W_1 \times J, W_1 = \left\{ u_1 \in \left( H^1 \left( \Omega_1 \right) \right)^d \text{ and } u_1 = u_1^0 \text{ on } \Gamma_1^D \right\} \\
\left\{ u_2 \left( X_2 , t \right) \right\} & \in W_2 \times J, W_2 = \left\{ u_2 \in \left( H^1 \left( \Omega_2 \right) \right)^d \text{ and } u_2 = u_2^0 \text{ on } \Gamma_2^D \right\}
\end{align*}
\]  

(Equation 24)

where \( d \) is the space dimension (equal to 1, 2 or 3). The introduction of the Lagrange multiplier field allows us to glue the velocities of the two subdomains at the interface \( \Gamma_I \) belonging to the dual space of displacements restricted to the interface.

All the above considered space variables are assumed to be sufficiently smooth and regular. Using a dual Schur formulation, the principle of virtual power for transient dynamics can be written. Find the solution \( u_1 \in W_1 \times J, u_2 \in W_2 \times J \) and \( \lambda \in Q \times J \), for which the following weak form is satisfied \( \forall \psi_3 \in W_1^* , \forall \psi_2 \in W_2^* \) (Equation 25):

\[
\begin{align*}
&\int_{\Omega_1} \rho_1 \psi_1 \cdot \ddot{u}_1 d\Omega + \int_{\Omega_1} \varepsilon \left( \psi_3 \right) : \sigma_1 d\Omega + \int_{\Omega_2} \rho_2 \psi_2 \cdot \ddot{u}_2 d\Omega + \int_{\Omega_2} \varepsilon \left( \psi_2 \right) : \sigma_2 d\Omega + \int_{\Omega_1} \rho_1 \psi_3 \cdot \ddot{u}_1 d\Omega = \int_{\Gamma} \psi_3 \cdot \left( b_1 - \dot{u}_1 \right) d\Gamma + \int_{\Gamma} \psi_2 \cdot \left( b_2 \right) d\Gamma + \int_{\Gamma} \psi_2 \cdot \left( \sigma_2 \right) \frac{\partial \psi_2}{\partial n} d\Gamma + \int_{\Gamma} \psi_2 \cdot \left( g_2^N \right) d\Gamma
\end{align*}
\]  

(Equation 25)

where the stress tensor \( \sigma_2 \) satisfies the behavior law given in Equation (2). Then, we follow the classical lines of the finite element discretization. At the interface between the subdomains, the continuity of velocities is imposed by the following condition (Equation 26):

\[
L_1 \dot{U}_1 + L_2 \dot{U}_2 = 0
\]  

(Equation 26)

where \( L_1 \) and \( L_2 \) are the Boolean matrices in the case of matching meshes at the interface; they operate on nodal vectors associated with the two subdomains \( \Omega_1 \) and \( \Omega_2 \); they pick out the degrees of freedom belonging to the interface \( \Gamma_I \) in order to ensure the kinematic continuity at the interface.

### 3.2 Time discretization of Hybrid Asynchronous absorbing layers

For the time discretization, the GC method proposed by Gravouil and Combescure (2002, 2001) is employed. Adopting the continuity of velocities at the interface, it was demonstrated that the coupling GC method is stable for any Newmark integrators (implicit and explicit) with their own time step depending on subdomains. As illustrated in Figure 3, an explicit time integrator with a fine time step \( \Delta t_1 \) is adopted for the subdomain \( \Omega_1 \) and an implicit time integrator with a large time step \( \Delta t_2 \) is used for subdomain \( \Omega_2 \), with \( \Delta t_2 = m\Delta t_1 \), \( m \) being the time step ratio between two subdomains. In this way, hybrid (different schemes associated) asynchronous (different time steps depending on subdomains) absorbing layers can be obtained. The
equilibrium of subdomain 2 is prescribed at time $t_m$ at the end of the large time $\Delta t_2$, while the equilibrium of subdomain 1 is prescribed at every time $t_j = j\Delta t_1$ ($j = 1, 2...m$) at the fine time scale. The gluing of the velocity at the interface is written at the fine time scale. Using the GC method, the wave propagation can be simulated using a fine time step, without being affected by the specific formulation adopted for the absorbing region at the boundary of the truncated mesh. Moreover, the multi-time step capabilities enable us to build efficient absorbing region in terms of computation time. Finally, the weak form given in Equation (25) with the velocity continuity equation in Equation (26) can be expressed in the following discrete form in space and time (Equations 27, 28, 29, 30 and 31):

$$M_1 \ddot{U}_1^j + K_1 U_1^j = F_1^{ext,j} - L_1^T \lambda^j \quad \text{at time } t=t_j$$

$$M_2 \ddot{U}_2^m + C_2 \dot{U}_2^m + C_2 U_2^m + K_2 U_2^m = F_2^{ext,m} - L_2^T \lambda^m \quad \text{at time } t=t_m$$

$$L_1 \dot{U}_1^j + L_2 \dot{U}_2^j = 0 \quad \text{at time } t=t_j$$

$$C_1 = \sum 2 \int_{\Omega} \rho \gamma \ [N^e]^T [N^e] d\Omega$$

$$C_2 = \sum \int_{\Omega} \rho \gamma^2 [N^e]^T [N^e] d\Omega$$

where $N^e$ is the shape function matrix in subdomain $\Omega_2$. The first equation is the discrete equation of motion of subdomains $\Omega_1$ written at time $t_j$ (fine time scale), whereas the second equation is the discrete equations of motion of subdomains $\Omega_2$ written at time $t_m$ (large time scale). On right hand side of the above equations, the interface forces enable the subdomains to be glued at their interface $I_i$. The third equation is the velocity continuity.

Newmark time integration schemes (1959) are adopted for the time discretization, characterized by the parameters $\gamma_2=0.5$ and $\beta_2=0.25$ for the implicit time integration (Constant Average Acceleration scheme) and the parameters $\gamma_1=0.5$ and $\beta_1=0$ for the explicit time integration scheme (Central Difference). By introducing the approximate Newmark formulae, it leads to the equations of motion written as (Equations 32 and 33):

$$\tilde{M}_1 \ddot{U}_1^j = F_1^{ext,j} - K_1 U_1^{j-1,p} - L_1^T \lambda^j$$

$$\tilde{M}_2 \ddot{U}_2^m = F_2^{ext,m} - C_1 \dot{U}_2^{0,p} - C_2 U_2^{0,p} - K_2 U_2^{0,p} - L_2^T \lambda^m$$

where $U_1^{j-1,p}$ and $\dot{U}_2^{0,p}$ denote the predictor values in terms of displacement and velocity, classically introduced through the approximate Newmark formulae; they correspond to quantities known at the beginning of the fine step and of the large time step, respectively.

The effective stiffness matrices $\tilde{M}_1$ and $\tilde{M}_2$ related to the two subdomains are defined by (Equations 34 and 35):

$$\tilde{M}_1 = M_1 + \beta_1 \Delta t_1^2 K_1$$

$$\tilde{M}_2 = M_2 + \beta_2 \Delta t_2^2 (K_2 + C_2) + \gamma_2 \Delta t_2 C_1$$

The kinematic quantities are divided into two parts: the free and the linked quantities in the coupling GC method. The free quantities are calculated by taking into account the internal and external forces, without considering the interface forces, whereas the linked quantities are obtained from the interface loads given by
It was demonstrated that the kinematic continuity condition can be expressed as a reduced-size interface problem as follows (Equation 36):

$$H\lambda^I = b_j$$  \hspace{1cm} (36)

with the interface operator and the right-hand side member vector defined by (Equation 37):

$$\left\{ \begin{array}{l}
H = \gamma_1 \Delta t L_1 \overline{M_1}^{-1} L_1^T + \gamma_2 \Delta t L_2 \overline{M_2}^{-1} L_2^T \\
b_j = L_1 U^\text{free},j + L_2 U^\text{free},j
\end{array} \right.$$  \hspace{1cm} (37)

The interface operator $H$ is called the Steklov-Poincaré operator which can be viewed as the condensed mass matrix on the degrees of freedom belonging to the interface between the two subdomains. The right hand-side vector $b_j$ only depends on the free velocities computed in both subdomains without considering the interface forces, it can be seen as a predictor value projected to the degrees of freedom belonging to the interface. Finally, once derived the Lagrange multiplier vector, the quantities related to the interface forces can be computed and the time step is completed by summing these linked quantities to the free quantities previously obtained.

4. EFFECTIVENESS OF HYBRID ASYNCHRONOUS ABSORBING LAYERS

In the section 2 and 3, in order to obtain the absorbing ability and no-reflection condition at the interface, the assumption of harmonic waves is applied. However, in reality, most of waves propagating are non-harmonic. On the basis of aforementioned formulas for designing absorbing layers, for the Rayleigh case, to define the absorbing ability and no-reflection condition, a frequency $f_0$ should be chosen. Similarly, though the absorbing ability of Kosloff is independent of frequency, the no-reflection condition at the interface requires a chosen frequency $f_0$. In the case of non-harmonic waves, if a frequency $f_0$ is defined, for the other frequencies, the attenuation coefficient $R_{\text{attenuation}}$ and the reflection coefficient $R_{\text{interface}}$ at the interface will not be the same as for $f_0$. Therefore, it is important to investigate the influence of the chosen frequency $f_0$ on the effectiveness of absorbing layer for non-harmonic waves. Next, Lamb’s test will be considered for assessing the effectiveness of the proposed hybrid asynchronous absorbing layers.

4.1 $R_{\text{interface}}$ and $R_{\text{attenuation}}$ for non-harmonic waves

In this part, related formulas are given below, because of the limit of paper length, the conclusion will be given directly. $R_{\text{interface}}$ of Kosloff layer for another frequency $f$, which is different from $f_0$, can be obtained as shown below (Equations 38 and 39):

$$R_{\text{interface}} = \frac{1 - \alpha^2 - \left( \frac{\alpha \omega}{\omega_0} \right)^2}{(1 + \alpha)^2 + \left( \frac{\alpha \omega}{\omega_0} \right)^2}$$  \hspace{1cm} (38)

$$\alpha = \frac{1}{\sqrt{1 + \left( \frac{f}{\omega_0} \right)^2}}$$  \hspace{1cm} (39)

Because of the independency of frequency, $R_{\text{attenuation}}$ of Kosloff layer is the same for all the frequencies. With regard to Rayleigh layer, the formula will be more complex, because in the demonstration (Zafati et al.
2014)[14], an assumption is applied: \( \frac{\alpha_M}{\omega_0} = \alpha_K \omega_0 = \xi \). The form of the complex wave number is changed by (Equation 40):

\[
k_p(\omega) = \frac{\omega}{V_{2p}} \sqrt{\frac{1 - \xi^2 - i \xi \left( \frac{\omega_0}{\omega} + \frac{\omega}{\omega_0} \right)}{1 + \xi^2 \left( \frac{\omega_0}{\omega} \right)^2}}
\]  

(Equation 40)

\( R_{interface} \) of Rayleigh layer for another frequency \( f \) is derived as below (Equations 41 and 42),

\[
R_{interface} = \frac{1 - e^{-\alpha 2 \xi} - i e^{-\alpha 2 \xi} \left( \frac{\omega_0}{\omega} - \frac{\omega}{\omega_0} \right)}{1 + e^{-\alpha 2 \xi} - i e^{-\alpha 2 \xi} \left( \frac{\omega_0}{\omega} - \frac{\omega}{\omega_0} \right)}
\]  

(Equation 41)

\[
\alpha = \frac{1}{\sqrt{1 + \xi^2}}
\]  

(Equation 42)

In terms of \( R_{attenuation} \) for Rayleigh layer, it also depends on frequency. The dimensionless imaginary part of \( k_p(\omega) \) which decides the decrement has been proved to decrease with increasing \( \frac{\omega_0}{\omega} \), it means that the high frequencies decay more quickly than the low frequencies. In other words, the relatively small \( f_0 \) can make more frequencies be attenuated over the \( R_{attenuation} \) defined by \( f_0 \). Hence, the relatively small \( f_0 \) is benefic for non-harmonic waves in terms of decrement. On the other hand, if the frequency \( f_0 \) is defined relatively small, reflection coefficient \( R_{interface} \) for other frequencies can be proved relatively smaller. Consequently, taking into account the benefits of the relatively small frequency \( f_0 \) in terms of decrement and interface, one method to reduce the reflection for the case of non-harmonic waves is to define the \( f_0 \) relatively small. Moreover, it can be proved that the influence of chosen frequency on the reflection for Kosloff is much less than for Rayleigh. For the chosen frequency which is much higher than the dominant frequency, due to the dependence of absorbing ability on the frequency in case of Rayleigh layer, there will be more frequencies which will decay less than the \( R_{attenuation} \) defined by \( f_0 \), it leads to the increase of reflection from the end of absorbing layer. For the chosen frequency which is much lower than the dominant frequency, it has been proved that it leads to the increase of reflection at the interface in case of Rayleigh layer, on the contrary, Kosloff is less sensitive than Rayleigh at the interface.

In brief, if the spectrum range of the incident wave is close to the dominant frequency, with an appropriate chosen frequency, the wave will penetrate the Rayleigh medium with small reflections from the interface and from the end. However, for the waves which have a large range of frequency or have different dominant frequencies, because the absorbing ability of Rayleigh medium depends on frequency and \( R_{interface} \) is not stable, it is difficult to find a chosen frequency \( f_0 \) to make the absorbing layer have a good behavior. In terms of absorbing ability, Kosloff is independent of frequency. For the reflection coefficient \( R_{interface} \), Kosloff is less sensitive than Rayleigh. Hence, Kosloff layer can better treat complex waves in comparison with Rayleigh layer.

### 4.2 Comparison with HA-PML

In this section, Lamb’s test is conducted by HA-Kosloff, HA-Rayleigh, HA-PML using a homogeneous time step in both subdomains, the soil is integrated in time with explicit scheme and the domain of absorbing layers with implicit scheme in order to avoid the critical time step in the soil partition to be affected by the
introduction of damping layers. As illustrated in Figure 4, the numerical model for three different absorbing layers is the same, composed of a bounded soil (subdomain 1) with a size of 250m and absorbing layers (subdomain 2) with the thickness of 250m. Lamb’s test has been simulated by Matlab using Explicit/Implicit co-simulation. The concentrated load applied to the surface of an infinite half space medium generates three types of waves propagating through the soil, involving P, S waves and Rayleigh waves (Lamb 1903). For this reason, Lamb’s test can be considered as a good test for assessing the performance of absorbing layer. Non-harmonic waves are investigated by considering a Ricker incident waves, the chosen values are: $t_p=3s$, $t_s=3s$ and $A =1MN$. The absorbing region parameters are: $R_{attenuation}$ equal to 0.01, $m$ equal to 2 on the basis of each general design formula. A point of observation is defined 20m from the loading on the surface.

![Figure 4. Lamb’s test: the soil is integrated in time with explicit scheme (CD) and absorbing layers with implicit scheme (CAA)](image)

![Figure 5. Vertical displacements at the observation point using different methods](image)
The horizontal and vertical displacements of three numerical models at the observation point are shown in Figures 5 and 6. We can observe that PML is the most precise, the reflected spurious wave is 0.27% in terms of the vertical displacement, 0.81% in terms of the horizontal displacement. In comparison, the reflected spurious waves of Kosloff and Rayleigh are in the same level with the chosen frequency $f_0$ equal to 0.1Hz. With respect of the vertical displacement, the reflected spurious wave is 1.38% for Kosloff and 1.51% for Rayleigh. With respect of the horizontal displacement, the reflected spurious wave is 0.94% for Kosloff and 1.15% for Rayleigh. It can be shown that the derived general design formulas can provide also good results in more general cases: 2D non-harmonic waves composed of body and surface waves.

The kinetic and internal energies of the soil domain are computed for different absorbing layers as shown in Figure 12. In order to distinguish the difference between the different results, the $L_2$ norm error is computed between energies for different absorbing layers and those of extended mesh (reference results). Considering a quantity $E$ over the time interval $[0, T]$, the $L_2$ norm is defined by (Equation 43):

$$\text{err} = \frac{\|E^{(m)} - E_{\text{ref}}\|_{L_2([0,T])}}{\|E_{\text{ref}}\|_{L_2([0,T])}}$$ (43)

$E^{(m)}$ is the kinetic or internal energy obtained by different absorbing layers and $E_{\text{ref}}$ is the reference energy obtained from the extended mesh. From Table 2, it can be observed that the errors are small for three types of absorbing layers. The errors of Kosloff and PML are clearly smaller than the errors of Rayleigh, because they are independent of frequency in terms of decrement.
Table 1. Relative errors of different methods

<table>
<thead>
<tr>
<th></th>
<th>Kinetic energy</th>
<th>Internal energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kosloff</td>
<td>0.12%</td>
<td>0.08%</td>
</tr>
<tr>
<td>Rayleigh</td>
<td>0.58%</td>
<td>0.25%</td>
</tr>
<tr>
<td>PML</td>
<td>0.14%</td>
<td>0.10%</td>
</tr>
</tbody>
</table>

The CPU times are recorded in a normalized form divided by CPU time of Kosloff layers with different time step ratios, as shown in Table 2. It shows that, for the same numerical model, Kosloff layers and Rayleigh layers use almost the same CPU time. PML is more complex, so it takes more time to calculate using the same numerical model. With time step ratio increasing, the CPU times of different absorbing layers decrease significantly. It implies that using explicit/implicit co-computation, not only the critical time step in the soil partition is not affected by the introduction of damping layer, but also large time steps can be adopted in absorbing layer domain to reduce the computation time.

Table 2. Normalized CPU Time for different methods using different time step ratios m.

<table>
<thead>
<tr>
<th></th>
<th>Kosloff</th>
<th>Rayleigh</th>
<th>PML</th>
</tr>
</thead>
<tbody>
<tr>
<td>m=1</td>
<td>1.00</td>
<td>1.02</td>
<td>1.39</td>
</tr>
<tr>
<td>m=5</td>
<td>0.44</td>
<td>0.46</td>
<td>0.70</td>
</tr>
<tr>
<td>m=10</td>
<td>0.36</td>
<td>0.40</td>
<td>0.58</td>
</tr>
</tbody>
</table>
5. CONCLUSION

In the present paper, two types of hybrid asynchronous absorbing layer are investigated in this paper for reproducing the elastic wave propagation in unbounded medium, which are Kosloff absorbing layer and Rayleigh absorbing layer. Firstly, the design of Kosloff absorbing layer is proposed by using the strong form of elastic wave propagation in Kosloff medium. The absorbing ability of Kosloff absorbing layer associated with a target performance criterion are derived in the form of logarithmic decrement and proved to be independent of frequency. In addition, no-reflection conditions are obtained to avoid the spurious waves reflected at the interface between physical domain and Kosloff absorbing layer domain. The general formulas for designing Kosloff absorbing layer using a multi-layer strategy are proposed. In the case of non-harmonic, Kosloff absorbing layer is proved to be independent of frequency in terms of attenuation and is less sensitive to the choice of the design parameters of the absorbing layer in comparison to Rayleigh absorbing layer.

Secondly, the GC method proposed by Gravouil and Combescure is employed by adopting the continuity of velocities at the interface to couple an independent explicit integrator for the subdomain of interest with a different implicit one for the absorbing layers based on the dual Schur approach. The derived absorbing region is called Hybrid (different time integrators) Asynchronous (different time steps) Absorbing Layers. The results of Lamb’s test shows that using explicit/implicit co-computation, not only the critical time step in the soil partition is not affected by the introduction of damping layer, but also large time steps can be adopted in absorbing layer domain to reduce the computation time. In comparison to the HA-PML, HA-Kosloff is easier to implement. Thanks to its independent frequency damping features, HA-Kosloff layers can be an alternative of PML to treat complex waves propagating problem.

6. REFERENCES


